Random Utility Models 1

Generic Random Utility Model

- Given a set of n options $A_1, A_2, ..., A_n$
  - Each option A_i is assigned a random utility, $U(A_i)$,
  - According to an n-dimensional density function.
- The probability of choosing A_i equals
  - $\Pr[A_i | \{A_1, ..., A_n\}] = \Pr\{U(A_i) = max[U(A_i), \ldots, U(A_n)]\}$.
  - same holds true for some arbitary subset
  - $\Pr\left[A_i | \left\{A_{j_1}, ..., A_{j_m}\right\}\right] = \Pr\{U(A_i) = max[U(A_{j_1}), \ldots, U(A_{j_m})]\}$ for $m < n$

General Properties of Random Utility theories

Regularity Satisfied

- Cannot increase choice probability for one option by adding other options to the set

$$
\begin{aligned}
& \Pr\left[A_i \middle| \left\{A_1, ...A_n, ...A_{n+m}\right\}\right] \\
= \;& \Pr\left[U_i = \max\left\{U_1, ...U_n\right\}\right] \\
& \times \Pr\left[U_i = \max\left\{U_1, ..., U_{n+m}\right\} \middle| U_i = \max\left\{U_1, ...U_n\right\}\right] \\
\leq \;& \Pr\left[U_i = \max\left\{U_1, ...U_n\right\}\right]
\end{aligned}
$$

- All random utility models satisfy Regularity

Triangle inequality satisfied (Regenwetter, 2010)

- $\Pr\left[A_i \middle| \{A_i, A_j\}\right] + \Pr\left[A_j \middle| \{A_j, A_k\}\right] \geq \Pr\left[A_i \middle| \{A_i, A_k\}\right]$

Proof of Triangle inequality

A	B	C	A>B	B>C	A>C
1	2	3	0	0	0
1	3	2	0	1	0
2	1	3	1	0	0
2	3	1	0	1	1
3	1	2	1	0	1
3	2	1	1	1	1

$$\Pr[A > B] + \Pr[B > C] = \Pr[A = 1, B = 3, C = 2]$$
$$+ \Pr[A = 2, B = 1, C = 3] + \Pr[A = 2, B = 3, C = 1]$$
$$+ \Pr[A = 3, B = 1, C = 2] + \Pr[A = 3, B = 2, C = 1]$$
$$\Pr[A > C] = \Pr[A = 3, B = 1, C = 2] + \Pr[A = 3, B = 2, C = 1]$$
$$+ \Pr[A = 2, B = 3, C = 1]$$

Weak Transitivity not satisfied by RUM's

- (See Tversky 1969; Regenwetter et al., 2010)

$$\Pr\left[A_i | \{A_i, A_j\}\right] \geq .50 \text{ and } \Pr\left[A_j | \{A_j, A_k\}\right] \geq .50$$
$$\rightarrow \Pr\left[A_i | \{A_i, A_k\}\right] \geq .50$$

Tversky 1969

	prob win	Amt
A	.60	100
B	.55	150
C	.50	100

- Lexico graphic rule produces violation of WST
  - First choose best on the basis of prob to win
  - If approximately equal on prob win, then choose best on basis of Amt to win
- Results
  - comparing A vs. B, difference in prob too small so choose B based on amount
  - comparing B vs. C, difference in prob too small so choose C based on amount
  - comparing A vs. C, difference in prob is large so choose A over C based on prob

Concordet paradox violates WST

- A person makes binary choices for all pairs at three time points using the utilities shown below

Utility table	A_1	A_2	A_3
T_1	3	2	1
T_2	1	3	2
T_3	2	1	3

A_1 beats A_2 at T_1 and T_3 so pooling across three times produces
$\Pr\left[A_1 \mid \{A_1, A_2\}\right] = 2/3$
A_2 beats A_3 at T_1 and T_2 so pooling across three times produces
$\Pr\left[A_2 \mid \{A_2, A_3\}\right] = 2/3$
A_3 beats A_1 at T_2 and T_3 so pooling across three times produces
$\Pr\left[A_3 \mid \{A_1, A_3\}\right] = 2/3$

Independence of irrelevant alternatives not satisfied

$$\begin{array}{rcl} \Pr\left[A_i \middle| \{A_1, ..A_n\}\right] & \geq & \Pr\left[A_j \middle| \{A_1, ..A_n\}\right] \\ & \rightarrow & \\ \Pr\left[A_i \middle| \{A_1, ..A_n, A_{n+1}, ..., A_{n+m}\}\right] & \geq & \Pr\left[A_j \middle| \{A_1, ..A_n, A_{n+1}, ..., A_{n+m}\}\right] \end{array}$$

- Some random utility models can violate this property

Thurstone Utility Model for Binary Choices (Probit model)

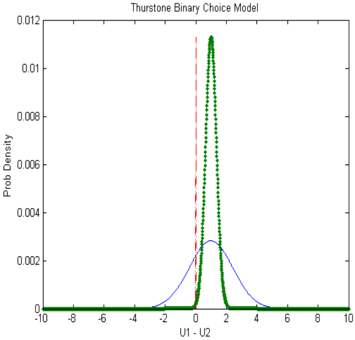
- Given a set of 2 options A_i, A_j
  - Each option A_i is assigned a random utility, $U(A_i)$,
  - According to a normal distribution with mean $\mu_i = E[U(A_i)]$
  - Variance $\sigma_i^2 = Var[U(A_i)] = E\left[(U(A_i) - \mu_i)^2\right]$
  - Covariance $\sigma_{ij} = Cov[U(A_i), U(A_j)] = E\left[(U(A_i) - \mu_i) \cdot \left(U(A_j) - \mu_j\right)\right]$
  - Correlation $\rho_{ij} = \sigma_{ij} / \left(\sigma_i \cdot \sigma_j\right)$

Choice probability for Binary Thurstone Utility Model

- The probability of choosing A_i over A_j equals

$$\begin{aligned}
\Pr\{U(A_i) &= max[U(A_i), U(A_j)]\} \\
&= \Pr\{U(A_i) - U(A_j) > 0\} \\
V &= U(A_i) - U(A_j) \\
& V\tilde{}N(\mu_V, \sigma_V^2) \\
\mu_V &= \mu_i - \mu_j \\
\sigma_V^2 &= \sigma_i^2 + \sigma_j^2 - 2 \cdot \sigma_i \cdot \sigma_j \cdot \rho_{ij} \\
\Pr[A_i | \{A_i, A_j\}] &= \int_{x>0} N(x) \cdot dx \\
&= F\left[\frac{\mu_V}{\sigma_V}\right] \\
\mu_V &> 0 \rightarrow \Pr[A_i | \{A_i, A_j\}] > .50
\end{aligned}$$

Thurstone Model for Binary Choice



- Two choice pairs, same mean difference but with different variances of difference
- Choice prob = area above red line
- Blue pair of choices
- $E[V] = 1, \sigma = 2, \Pr[A_1 | \{A_1, A_2\}] = .6915$
- Green pair of choices
- $E[V] = 1, \sigma = 1/2, \Pr[A_1 | \{A_1, A_2\}] = .9772$

Properties of Thurstone Model for Binary Choice

- **Weak** Stochastic Transitivity is satisfied
  - $\Pr\left[A_i|\left\{A_i, A_j\right\}\right] \geq .50$ implies $\mu_i \geq \mu_j$
  - $\Pr\left[A_j|\left\{A_j, A_k\right\}\right] \geq .50$ implies $\mu_j \geq \mu_k$
  - $\mu_i \geq \mu_j \geq \mu_k$ implies $\Pr\left[A_i|\left\{A_i, A_k\right\}\right] \geq .50$
- It also obeys **moderate** stochastic Transitivity (Halff, JMP, 1976)
  - $\Pr\left[A_i|\left\{A_i, A_j\right\}\right] \geq .50$ and $\Pr\left[A_j|\left\{A_j, A_k\right\}\right] \geq .50$ implies
  - $\Pr\left[A_i|\left\{A_i, A_k\right\}\right] \geq \min\left\{\Pr\left[A_i|\left\{A_i, A_j\right\}\right], \Pr\left[A_j|\left\{A_j, A_k\right\}\right]\right\}$

Random weight model

$$
\begin{aligned}
U_i &= W_Q \cdot u(Q_i) + W_E \cdot u(E_i) \\
& \quad W_Q \tilde{} N(w_Q, \sigma_Q^2) \\
& \quad W_E \tilde{} N(w_E, \sigma_E^2) \\
Cov(W_Q, W_E) &= 0 \\
V &= U_i - U_j \\
E[V] &= w_Q \cdot [u(Q_i) - u(Q_j)] + w_E \cdot [u(E_i) - u(E_j)] \\
Var[V] &= \sigma_Q^2 \cdot [u(Q_i) - u(Q_j)]^2 + \sigma_E^2 \cdot [u(E_i) - u(E_j)]^2
\end{aligned}
$$

Thurstone Case V

- Assume $\sigma_i^2 + \sigma_j^2 - 2 \cdot \sigma_i \cdot \sigma_j \cdot \rho_{ij} = \sigma^2$
- **Constant** variance of difference for all pairs
- $\Pr[A_i | \{A_i, A_j\}] = F\left[\left(\mu_i - \mu_j\right)\right]$
  - Strictly increasing monotonic function of mean difference

Thurstone Case V obeys Strong Stoch Transitivity

$$
\begin{aligned}
\Pr\left[A_i | \{A_i, A_j\}\right] &\geq .50 \rightarrow \mu_i \geq \mu_j \\
\Pr\left[A_j | \{A_j, A_k\}\right] &\geq .50 \rightarrow \mu_j \geq \mu_k \\
\mu_i &\geq \mu_j \geq \mu_k \\
\mu_i - \mu_k &\geq \mu_i - \mu_j \rightarrow \Pr\left[A_i | \{A_i, A_k\}\right] \geq \Pr\left[A_i | \{A_i, A_j\}\right] \\
\mu_i - \mu_k &\geq \mu_j - \mu_k \rightarrow \Pr\left[A_i | \{A_i, A_k\}\right] \geq \Pr\left[A_j | \{A_j, A_k\}\right] \\
\Pr\left[A_i | \{A_i, A_k\}\right] &\geq \max\left\{\Pr\left[A_i | \{A_i, A_j\}\right], \Pr\left[A_j | \{A_j, A_k\}\right]\right\}
\end{aligned}
$$

Humans violate SST (Mellers and Biagini, 1994, Psychological Review)

Laptop	wgt	cost
X	1.03 kg	$1000
Y	1.02 kg	$1100
Z	.45 kg	$2000

$$\Pr[X|\{X,Y\}] > \Pr[X|\{X,Z\}] > \Pr[Y|\{Y,Z\}] > .50$$

Violates SST

Correlation changes across pairs

Need general Thurstone model

Thurstone Case V obeys Independence

$$\begin{aligned}
\Pr\left[A_i \middle| \left\{A_i, A_k\right\}\right] &\geq \Pr\left[A_j \middle| \left\{A_j, A_k\right\}\right] \rightarrow \mu_i - \mu_k \geq \mu_j - \mu_k \\
\mu_i - \mu_k &\geq \mu_j - \mu_k \rightarrow \mu_i - \mu_l \geq \mu_j - \mu_l \\
\mu_i - \mu_l &\geq \mu_j - \mu_l \rightarrow \Pr\left[A_i \middle| \left\{A_i, A_l\right\}\right] \geq \Pr\left[A_j \middle| \left\{A_j, A_l\right\}\right]
\end{aligned}$$

Humans violate IIA (Busemeyer & Townsend, 1993, Psychological Review)

Action	H	T
X	$1.00	−$1.00
Y	$.02	−$.02
Z	$.01	$.01
W	−$.01	−$.01

$$\Pr[X | \{X, Z\}] > \Pr[Y | \{Y, Z\}]$$

$$\Pr[X | \{X, W\}] < \Pr[Y | \{Y, W\}]$$

Variance changes across pairs

Need to use general Thurstone model

Thurstone Model for Multiple Choice

- $n-$ alternatives
- $\mathbf{U}_n$ random vector of utilities
- $\mathbf{U}_n \tilde{} N(\mu_n, \Sigma)$
- $\boldsymbol{\mu}_n = E[U_n]$ $n \times 1$ centroid
- $Var[\mathbf{U}_n] = \Sigma = E\left[(\mathbf{U}_n - \boldsymbol{\mu}_n) \cdot (\mathbf{U}_n - \boldsymbol{\mu}_n)'\right]$ $n \times n$ variance - covariance matrix

Choice Probabilities for Multiple Choice

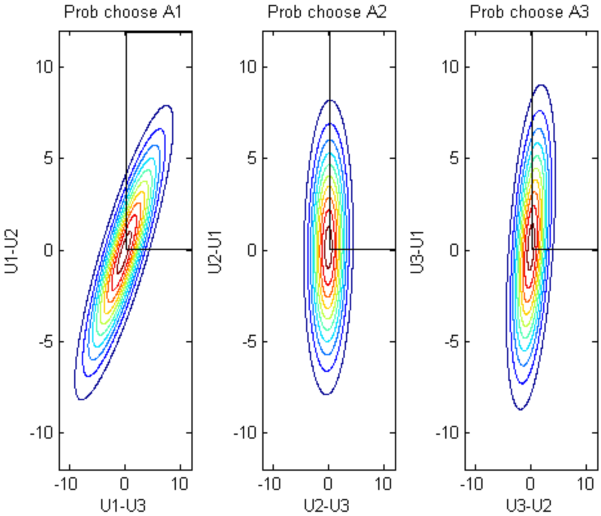
- E.g. $n = 3$

$$\Pr\left[A_j \middle| \{A_i, A_j, A_k\}\right]$$
$$= \Pr[U_j - U_i > 0, U_j - U_k > 0]$$

$$\begin{aligned}
U &= \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}, \; L = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \\
V &= L \cdot U = \begin{bmatrix} U_2 - U_1 \\ U_2 - U_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\
\Pr\left[A_j \middle| \{A_i, A_j, A_k\}\right] &= \Pr\left[V_1 > 0, V_2 > 0\right]
\end{aligned}$$

Choice Probabilities for Multiple Choice

- E.g. $n = 3$



Choice Probabilities for Multiple Choice

$$X = \begin{bmatrix} .1 & .9 \\ .8 & .2 \\ .9 & .1 \end{bmatrix},\ w = \begin{bmatrix} .6 \\ .4 \end{bmatrix},\ X \cdot w = \begin{bmatrix} .42 \\ .56 \\ .58 \end{bmatrix}$$

$$A_1 : E[V] = \begin{bmatrix} -.14 \\ -.16 \end{bmatrix},\ var(V) = \begin{bmatrix} 13.8 & 13.2 \\ 13.2 & 16.8 \end{bmatrix},$$

$$A_2 : E[V] = \begin{bmatrix} .14 \\ -.02 \end{bmatrix},\ var(V) = \begin{bmatrix} 13.8 & .60 \\ .6 & 4.2 \end{bmatrix},$$

$$A_3 : E[V] = \begin{bmatrix} .16 \\ .02 \end{bmatrix},\ var(V) = \begin{bmatrix} 16.8 & 3.6 \\ 3.6 & 4.2 \end{bmatrix},$$

$$\Pr[A_1 | \{A_1, A_2\}] = .48,\ \Pr[A_1 | \{A_1, A_3\}] = .47,\ \Pr[A_2 | \{A_2, A_3\}] = .49$$

$$\Pr[A_1] = .40,\ \Pr[A_2] = .27,\ \Pr[A_3] = .33$$

Random Coefficient Model

- $U_i = \sum_{j=1}^{p} w_j \cdot s_{ij}$
- s_{ij} := scale value of attribute j for alternative i
- w_j := random weight given to attribute j
- W := $p \times 1$ vector of random weights
- S := $n \times p$ matrix of scale values (rows are alternatives, columns are attributes)
- $U = S \cdot W$:= $n \times 1$ vector of random utilities

Choice probability for Random Coefficient Model

$$
\begin{aligned}
& W\tilde{}N(\mathbf{w},\Psi),\ p\times 1 \\
& U\tilde{}N(\boldsymbol{\mu},\Sigma),\ n\times 1 \\
\boldsymbol{\mu} &= S\cdot\mathbf{w},\ p\times 1 \\
\Sigma &= S\cdot\Psi\cdot S',\ p\times p \\
V &= L\cdot U,\ (n-1)\times 1 \\
& V\tilde{}N(\boldsymbol{v},\Phi) \\
\boldsymbol{v} &= L\cdot\boldsymbol{\mu},\ (n-1)\times 1 \\
\Phi &= L\cdot\Sigma\cdot L',\ (n-1)\times(n-1) \\
\Pr\left[A_i \middle| \{A_1,...,A_n\}\right] &= \Pr\left[V_1>0,...,V_{n-1}>0\right]
\end{aligned}
$$