Random Utility Models 2

Luce Choice Model

- Given a set of n options $A_1, A_2, ..., A_n$
  - Each option A_i is assigned a fixed utility, $u(A_i)$
  - $v(A_i) = \exp(u(A_i))$ is the strength of option i
- The probability of choosing A_i equals
  - $\Pr[A_i | \{A_1, ..., A_n\}] = \frac{v(A_i)}{\sum_{j=1}^{n} v(A_j)}$
  - same holds true for some arbitary subset
  - $\Pr[A_{j_i} | \{A_{j_1}, ..., A_{j_m}\}] = \frac{v(A_{j_i})}{\sum_{k=1}^{m} v(A_{j_k})}$ for $m < n$

Relation between Luce and Logistic binary choice models

$$\begin{aligned}
\Pr\left[A \mid \{A, B\}\right] &= \frac{v(A)}{v(A) + v(B)} \\
&= \frac{e^{u(A)}}{e^{u(A)} + e^{u(B)}} \\
&= \frac{e^{-u(A)}}{e^{-u(A)}} \cdot \frac{e^{u(A)}}{e^{u(A)} + e^{u(B)}} \\
&= \frac{1}{1 + e^{-[u(A) - u(B)]}}
\end{aligned}$$

Logistic Choice model

$$\frac{\Pr\left[A \mid \{A,B\}\right]}{\Pr\left[B \mid \{A,B\}\right]} = \frac{v(A)}{v(B)} = \exp\left(u(A) - u(B)\right)$$

$$\log\frac{\Pr\left[A \mid \{A,B\}\right]}{\Pr\left[B \mid \{A,B\}\right]} = u(A) - u(B)$$

$$u(A) = \sum_{j=1}^{p} w_j \cdot s_{Aj}$$

$$u(B) = \sum_{j=1}^{p} w_j \cdot s_{Bj}$$

Properties of Luce Choice model

Regularity property

$$\begin{aligned} \Pr\left[A_i \middle| \{A_1, ..., A_n\}\right] &= \frac{v(A_i)}{\sum_{j=1}^{n} v(A_j)} \geq \frac{v(A_i)}{\sum_{j=1}^{n} v(A_j) + \sum_{j=n+1}^{n+m} v(A_j)} \\ &= \Pr\left[A_i \middle| \{A_1, ..., A_n, ..A_{n+m}\}\right] \end{aligned}$$

SST property

$$
\begin{aligned}
\Pr[A|\{A,B\}] &\geq .50 \rightarrow v(A) \geq v(B) \\
\Pr[B|\{B,C\}] &\geq .50 \rightarrow v(B) \geq v(C) \\
v(A) &\geq v(B) \geq v(C) \\
\Pr[A|\{B,C\}] &\geq \max\left[\Pr[A|\{A,B\}], \Pr[B|\{B,C\}]\right]
\end{aligned}
$$

Luce's choice axiom: Strong form of IIA

$$
\begin{aligned}
& \Pr\left[A_i \middle| \left\{A_1, ...A_n, ...A_{n+m}\right\}\right] \\
= \quad & \frac{v\left(A_i\right)}{\sum\limits_{j=1}^{n+m} v\left(A_j\right)} = \frac{v\left(A_i\right)}{\sum\limits_{j=1}^{n} v\left(A_j\right)} \cdot \frac{\sum\limits_{j=1}^{n} v\left(A_j\right)}{\sum\limits_{j=1}^{n+m} v\left(A_j\right)} \\
= \quad & \Pr\left[A_i \middle| \left\{A_1, ..., A_n\right\}\right] \\
& \times \Pr\left[\left\{A_1, ..., A_n\right\} \middle| \left\{A_1, ...A_n, ...A_{n+m}\right\}\right]
\end{aligned}
$$

Product Rule

$$
\begin{aligned}
& \frac{\Pr\left[A|\left\{A,B\right\}\right]}{\Pr\left[B|\left\{A,B\right\}\right]} \cdot \frac{\Pr\left[B|\left\{B,C\right\}\right]}{\Pr\left[C|\left\{C,B\right\}\right]} \\
= & \frac{v\left(A\right)}{v\left(B\right)} \cdot \frac{v\left(B\right)}{v\left(C\right)} = \frac{v\left(A\right)}{v\left(C\right)} \\
= & \frac{\Pr\left[A|\left\{A,C\right\}\right]}{\Pr\left[C|\left\{A,C\right\}\right]}.
\end{aligned}
$$

Similarity Effects violate Luce Choice axiom

- Paris Plus a dollar (Debreu)
  - Choose between trip to Rome versus Trip to Paris (.50)
  - Choose between trip to Paris versus Paris plus one dollar (1.0)
  - Choose between trip to Rome versus Paris plus one dollar (.50)
- Red bus, blue bus (McFadden)
  - Choose between Red bus and car to go to work (.50)
  - Choose between the Blue bus and car to go to work (.50)
  - Choose between the Red bus or Blue bus or car to go to work (.25, .25, .50)

Extreme value random utility

- Suppose the random utility of option A_i has the following probability density function for non-positive values

$$f_i(u) = v_i \cdot e^{v_i \cdot u}, \; u \leq 0$$

- The cumulative distribution equals

$$\begin{aligned} \Pr[U \leq u] &= \int_{-\infty}^{u} v_i \cdot e^{v_i \cdot x} dx \\ &= e^{v_i \cdot x}\Big|_{-\infty}^{u} = \left(e^{v_i \cdot u} - e^{-\infty}\right) \\ &= e^{v_i \cdot u}. \end{aligned}$$

Deriving Luce model from Extreme values (Yellot, 1977)

- Suppose the random utilities are independent

$$
\begin{aligned}
& \Pr\left[U_i = \max\left\{U_1, U_2, ..., U_n\right\}\right] \\
= \quad & \int_{-\infty}^{0} f_i\left(u\right) \prod_{j \neq i} \Pr\left[U_j \leq u\right] \cdot du \\
= \quad & v_i \cdot \int_{-\infty}^{0} e^{v_i \cdot u} \cdot \prod_{j \neq i} e^{v_j \cdot u} \cdot du \\
= \quad & v_i \cdot \int_{-\infty}^{0} \exp\left(\sum_{j=1}^{n} v_j \cdot u\right) \cdot du \\
= \quad & \frac{v_i}{\sum_{j=1}^{n} v_j} \cdot \exp\left(\sum_{j=1}^{n} v_j \cdot u\right) \Big|_{-\infty}^{0} \\
= \quad & \frac{v_i}{\sum_{j=1}^{n} v_j}.
\end{aligned}
$$

McFadden Generalized extreme value model

- Assume that the complete choice set of n options can be divided into m groups, each is a group of similar options.
- C_k represents a subset of similar options that belong to group k.
- For example with a complete set $\mathrm{X} = \{\text{Red Bus, Blue Bus, Car}\}$ we could posit $C_1 = \{\text{Red Bus, Blue Bus}\}$, and $C_2 = \{\text{Car}\}$. In this example $\mathrm{m} = 2$.

$$\Pr\left[A_i \middle| \{A_1, ..., A_n\}\right] = \sum_{k=1}^{m} \Pr\left[C_k\right] \cdot \Pr\left[A_i \middle| C_k\right]$$

McFadden Generalized extreme value random utility model

$$
\begin{aligned}
\Pr\left[A_i \middle| C_k\right] &= \frac{e^{u(A_i)/\theta_k}}{\sum_{j \in C_k} e^{u(A_j)/\theta_k}}, \text{ for } A_i \in C_k \\
h_l &= \ln\left(\sum_{j \in C_l} e^{u(A_i)/\theta_l}\right) \\
\Pr\left[C_k\right] &= \frac{a_k \cdot e^{\theta_k \cdot h_k}}{\sum_{l=1}^{m} a_l \cdot e^{\theta_l \cdot h_l}}, \; a_l \geq 0
\end{aligned}
$$

Application to X = {red bus, blue bus, car} problem

$$\Pr\left[\text{Car}\middle|\left\{\text{Car,Rbus}\right\}\right] = \frac{\exp\left(\frac{u(Car)}{\theta_{C,R}}\right)}{\exp\left(\frac{u(Car)}{\theta_{C,Bus}}\right) + \exp\left(\frac{u(Rbus)}{\theta_{C,Bus}}\right)}$$

$$\Pr\left[\text{Car}\middle|\left\{\text{Car,Bbus}\right\}\right] = \frac{\exp\left(\frac{u(Car)}{\theta_{C,B}}\right)}{\exp\left(\frac{u(Car)}{\theta_{C,Bus}}\right) + \exp\left(\frac{u(Bbus)}{\theta_{C,Bus}}\right)}$$

$$\Pr\left[\text{Rbus}|\text{Bus}\right] = \frac{\exp\left(\frac{u(Rbus)}{\theta_{Bus}}\right)}{\exp\left(\frac{u(Rbus)}{\theta_{Bus}}\right) + \exp\left(\frac{U(Bbus)}{\theta_{Bus}}\right)}$$

Application to X = {red bus, blue bus, car} problem

$$h_{Car} = \ln\left(\exp\left(\frac{u(\text{Car})}{\theta_{Car}}\right)\right)$$

$$h_{Bus} = \ln\left(\exp\left(\frac{u(Rbus)}{\theta_{Bus}}\right) + \exp\left(\frac{u(Bbus)}{\theta_{Bus}}\right)\right)$$

$$\Pr[\text{C}|\text{X}] = Q_C = \frac{a_C \cdot \exp(h_{Car} \cdot \theta_{Car})}{a_C \cdot \exp(h_{Car} \cdot \theta_{Car}) + a_B \cdot \exp(h_{Bus} \cdot \theta_{Bus})}$$

$$\begin{aligned} Q_B &= \frac{a_B \cdot \exp(h_{Bus} \cdot \theta_B)}{a_C \cdot \exp(h_{Car} \cdot \theta_{Car}) + a_B \cdot \exp(h_{Bus} \cdot \theta_{Bus})} \\ \Pr[\text{Rbus}|\text{X}] &= Q_B \cdot \Pr[\text{Rbus}|\text{Bus}] \end{aligned}$$

McFadden Generalized extreme value utility model

- satisfies regularity
- can violate SST
- can violate IIR
- satisfied triangular inequality

Mixed Logit Model

- Define $L(A_i|w)$ as the probability that option A_i is chosen from a set given a fixed set of weight coefficients w. In particular, L can be defined by the Logistic model.

$$
\begin{aligned}
L(A_i|w) &= \frac{e^{u(A_i)}}{\sum e^{u(A_i)}}, \\
u(A_i) &= \sum w_j \cdot s_{ij}
\end{aligned}
$$

- Now suppose that $\Pr(A_i)$ is given by a probability mixture of $L(A_i|w)$ as defined by the integral over the density

$$\Pr[A_i] = \int f(w) \cdot L(A_i|w) \cdot dw$$

Mixed Logit Model

- Can violate WST
- Can violate IIR
- satisfies regularity