Analysis of Multiplicative Combination Rules When the Causal Variables are Measured With Error

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A variety of theories in psychology postulate that the causal variables combine according to a multiplicative rule to determine the value of the dependent variable. To test multiplicative combination rules empirically, applied researchers frequently use an observational method that involves the following procedure: (a) assessment techniques are used to measure the value of each theoretical construct for each individual, (b) product scores are formed by multiplying the measures of the causal variables, and (c) hierarchical regression analysis is used to test the statistical significance of the increment in R^2 contributed by the product term. The purpose of this article is to evaluate the validity of the observational method with respect to two measurement issues: measurement level (i.e., the effects produced by allowing monotonic transformations of the measures), and measurement error (i.e., the effects produced by using unreliable measures of the causal variables). Our evaluation is based on a theoretical distinction between the structural model (the set of equations relating theoretical constructs to each other) and the measurement model (the set of equations relating the theoretical constructs to the observed measures). We conclude that hierarchical regression analysis is inadequate for determining whether the structural model is additive or multiplicative for two reasons. First, an additive structural model may produce multiplicative effects through a nonlinear measurement model. Second, a multiplicative structural model may produce nondetectable multiplicative effects because of multiplicative measurement error. Some alternatives to hierarchical regression analysis are described.

Multiplicative models are ubiquitous in psychological research. A famous example from learning theory is Hull's (1943) model: Response strength is a function of Drive × Habit. Another classic example is Edwards' (1954) model of decision making: Utility of a gamble equals Subjective Probability × Utility of each outcome, summed across outcomes. Two important examples from industrial/organizational psychology are Vroom's (1964) models: Motivation to work is a function of Expectancy × Valence, summed across outcomes, and performance is a function of Ability × Motivation. Finally, an example from attitude research and consumer psychology is Fishbein and Ajzen's (1975) model: Attitude equals Belief × Evaluation, summed across salient beliefs.

This widespread use of multiplicative models in psychology indicates the need for rigorous methods to empirically evaluate their validity. Black (1965) has reviewed some experimental methods that were designed explicitly to evaluate additive versus multiplicative rules generated from Hull-Spence theories.[1] More recently, Anderson and Shan-

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[1] Several recent measurement theorists have described ordinal tests of Hull-Spence models. However, a close

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teau (1970) have developed tests of the subjective expected utility model based on functional measurement methods. Krantz and Tversky (1971) have described methods based on ordinal comparisons for diagnosing various composition rules. Finally, Birnbaum (1982) has described "scale free" and "scale convergence" methods for testing ratio and difference models of judgment. All of these are considered experimental methods because they require manipulation of the independent variables according to special experimental designs and they use specific contrasts among cell means to test the multiplicative model against the alternative of simple additivity.

Unfortunately, it is often difficult or impossible for researchers to perform the manipulations of theoretical variables required by the experimental method. This is especially true of individual difference variables such as ability. As a result, many researchers working in attitude theory (see Fishbein & Ajzen, 1975, chap. 3), work motivation theory (see Vroom, 1964, chap. 2), and personality theory (see Wiggins, 1973, chap. 3) have resorted to an observational method. The methodology can be summarized as follows: (a) a sample of individuals is selected for study; (b) assessment techniques (e.g., personality tests, aptitude tests, or attitude scales) are used to measure the value of each variable for each individual; (c) product scores are computed from the measured values of the variables; and (d) multiple regression analysis is used to evaluate the multiplicative model.

The purpose of the present article is to evaluate the usefulness of the observational method for empirically testing multiplicative theories. Two major issues threaten the validity of theoretical conclusions based on the observational method—the problem of measurement level and the problem of measurement error. A more precise definition of these two issues is possible by considering an example.

reading of Spence's theory would indicate that the Spence model is not a simple polynomial. As Black (1965) pointed out earlier, the intervening variable incentive is a classically conditioned response, and this learning process is a function of consummatory behavior. Thus, incentive and drive are functionally related.

An Example: Performance = Ability × Motivation

Suppose a researcher was interested in testing Vroom's theory that Performance = Ability × Motivation using the observational method. Figure 1 is a causal diagram that is useful for illustrating the theoretical issues. Referring to Figure 1, the uppercase letters represent theoretical constructs. In particular, define the letters A, M, and P as A = ability, M = motivation, and P = performance potential. These constructs are latent random variables representing individual differences and are not directly observable. The symbol $\rho_{A,M}$ represents the correlation between A and M. The subscripted variables X_a and X_m are fallible measures of the corresponding constructs A and M. For example, define X_a and X_m as X_a = ability test score, and X_m = effort rating. The subscripted variables Y_{p_1} and Y_{p_2} are both fallible measures of the performance construct P. For example, define Y_{p_1} and Y_{p_2} as Y_{p_1} = response latency, and Y_{p_2} = response accuracy. The two measures Y_{p_1} and Y_{p_2} illustrate the idea that any given construct may have multiple indicators. For convenience, the subscripted variable Y_p will be used to refer to either Y_{p_1} or Y_{p_2}, whenever it is not important to distinguish these two measures. These measures are assumed to be related to the theoretical constructs by the following system of equations, which will be referred to collectively as the measurement model:

$$X_a = F(A) + E_a, \tag{1a}$$

$$X_m = G(M) + E_m, \tag{1b}$$

$$Y_p = H(P) + E_p, \tag{1c}$$

where F, G, and H are monotonic functions, and E_a, E_m, and E_p are random measurement errors. The exact form of the monotonic functions F, G, and H may be known, partially known, or unknown. For convenience, the variables X_a and X_m will be referred to as predictor variables because they are measures of the hypothesized causal variables A and M, and the observed measure Y_p will be referred to as the criterion variable because it is a measure of the hypothesized dependent variable P.

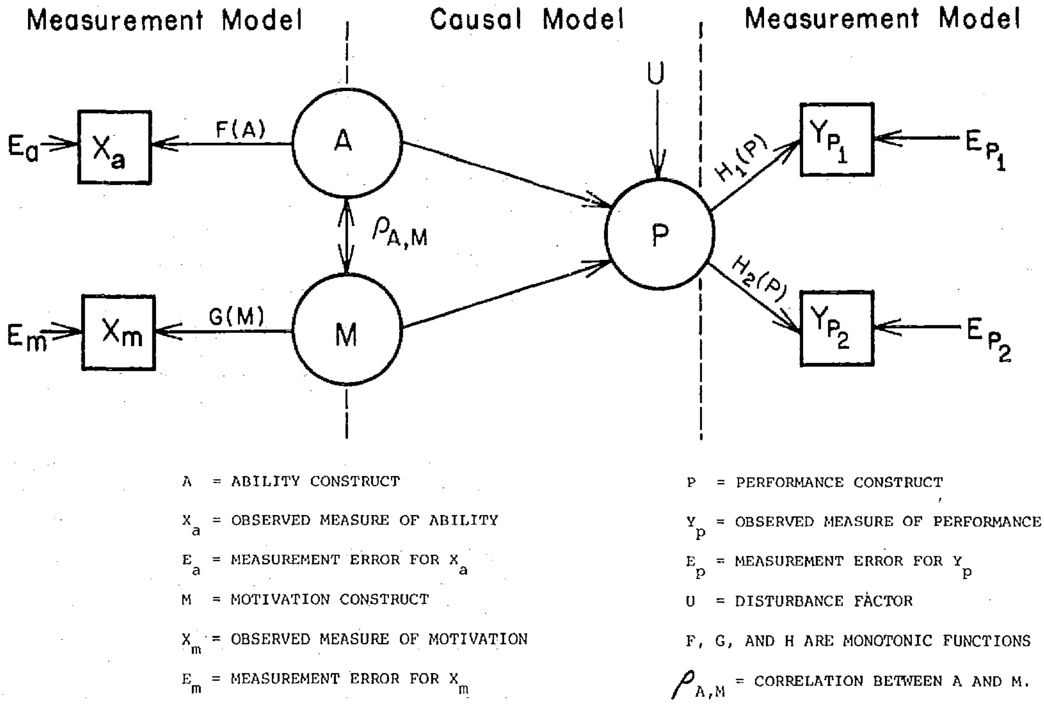

Figure 1. A causal diagram of Vroom's model of performance, with multiple indicators of one construct.

The paths in Figure 1 showing the influence of A and M on P represent the hypothesized causal relations, and the symbol U represents unknown causal factors that also influence P. The unknown disturbance U is assumed to be uncorrelated with all other causal variables. The set of functional relations among the theoretical constructs is referred to as the structural model (cf. Rock, Werts, Linn, & Joreskog, 1977). The structural multiplicative model can be stated as

$$P = B_0 + B_1 \cdot A + B_2 \cdot M + B_3 \cdot (M \cdot A) + U, \quad (2)$$

where $B' = (B_0, B_1, B_2, B_3)$ are called the structural parameters. The additive structural model can be obtained by assuming that $B_3 = 0$.

The goal of the researcher is to determine whether the structural model is additive or multiplicative. Because the true structural relations are not directly observable, they must be inferred from statistical relations among the observed measures. Since the estimated regression equation is influenced by both the measurement model and the underlying structural model, any assumptions concerning the measurement model must be carefully considered so that valid inferences about the structural model can be made.

It is now possible to state the issues regarding measurement level and measurement error more precisely. In the section on the measurement level problem, we analyze the effect that restrictions on the forms of the functions F, G, and H have on empirical tests of the structural model. In the section on the measurement error problem, we analyze the effect of assumptions concerning the random errors E_a, E_m, and E_p on empirical tests of the structural model.

The Measurement Level Problem

To simplify the analysis of the measurement level problem, it will be assumed that there is no measurement error (i.e., $E_a = E_m = E_p = 0$). The effects of measurement error will be considered in a later section. Table 1 breaks the measurement level problem down into four categories.

Table 1
Four Cases of the Measurement Level Problem

Predictors X_a, X_m	Criterion Y_p: Linear	Criterion Y_p: Monotonic
Linear	F, G, and H all linear[a]	H monotonic[b] F, G, linear
Monotonic	H linear F, G, monotonic	F, G, and H monotonic

[a] Linear function: $F(X) = C_0 + C_1X$, where C_0 and C_1 are unknown scale constants.
[b] Monotonic functions:

$$F(X) = C_0 + C_1\log(X).$$

$$F(X) = C_0 + C_1\exp(X).$$

$$F(X) = C_0 + C_1X + C_2X^2, C_1, C_2,$$

restricted to maintain monotonicity.

Any function $F(X)$ such that $\dfrac{\partial F(X)}{\partial X} > 0, \dfrac{\partial^2 F(X)}{\partial^2 X} < 0.$

Category 1: Linear Predictors and Criterion

In this category, $X_a = F(A) = C_1 + C_2 \cdot A$, $X_m = G(M) = C_3 + C_4 \cdot M$, and $Y_p = H(P) = C_5 + C_6 \cdot P$, where the Cs are unknown constants representing the origins and the units of the scales. The effect of these scale constants is to bias the estimates of the structural parameters. This bias is revealed by writing the structural equation in terms of the observed measures. Substituting $P = (Y_p - C_5)/C_6$, $M = (X_m - C_3)/C_4$, and $A = (X_a - C_1)/C_2$ into Equation 2 yields

$$\begin{aligned}(Y_p - C_5)/C_6 &= B_0 + B_1 \cdot (X_a - C_1)/C_2 \\ &+ B_2 \cdot (X_m - C_3)/C_4 \\ &+ B_3 \cdot (X_a - C_1)(X_m - C_3)/(C_2 \cdot C_4) + U,\end{aligned}$$

which reduces to the following multiple regression model:

$$Y_p = b_0 + b_1 \cdot X_a + b_2 \cdot X_m + b_3 \cdot (X_a \cdot X_m) + e, \quad (3)$$

where

$$b_1 = (C_6/C_2) \cdot [B_1 - (B_3 \cdot C_3/C_4)].$$

$$b_2 = (C_6/C_4) \cdot [B_2 - (B_3 \cdot C_1/C_2)],$$

$$b_3 = (B_3 \cdot C_6)/(C_2 \cdot C_4),$$

and

$$e = C_6 \cdot U.$$

Equation 3 implies that even when the structural model is assumed to be strictly multiplicative (i.e., $B_1 = B_2 = 0$, $B_3 \neq 0$ in Equation 2), the regression model could contain both additive and multiplicative effects (i.e., b_1, b_2, and b_3 are all nonzero) because of the additive constants C_1 and C_3. However, if the structural model is strictly additive (i.e., $B_3 = 0$ in Equation 2), this implies that the regression model will also be strictly additive (i.e., $b_3 = 0$ in Equation 3).

Cohen (1968, 1978) has proposed a test of the additive versus the multiplicative regression model, based on hierarchical regression analysis, which is unaffected by the scales of the measures in Equation 3. The hierarchical regression test is performed by comparing the percentage of variance predicted by Equation 3, with no restrictions on the parameters (symbolized as R^2_{mult}), with the percentage of variance predicted by Equation 3, with the restriction that $b_3 = 0$ (symbolized as R^2_{add}). The increment in R^2, $\Delta R^2 = R^2_{\text{mult}} - R^2_{\text{add}}$, provides the test statistic for testing the significance of the Linear × Linear trend component in Equation 3. The value of ΔR^2 will be invariant to linear transformations of the scales of X_a and X_m.

Cohen's work was primarily concerned with data analytic issues. Arnold and Evans (1979) have gone further and argued that hierarchical regression analysis can be used to empirically evaluate psychological theories, and such techniques have been applied to theories of work motivation by Arnold and House (1980). We argue below that there are a number of complex issues that may limit the usefulness of hierarchical regression analysis for empirically evaluating psychological theories.

Category 2: Linear Criterion, Monotonic Predictors

In this category, $Y_p = H(P) = C_5 + C_6 \cdot P$, but $X_a = F(A)$ and $X_m = G(M)$ are unknown or partially known monotonic functions. In order to provide concrete examples, it will be useful to assume that the inverse functions $A = F^{-1}(X_a)$ and $M = G^{-1}(X_m)$ can be approximated by quadratic functions. Under these assumptions, it is possible to write the structural model in terms of the observable measures as

$$(Y_p - C_5)/C_6 = B_0 + B_1 \cdot F^{-1}(X_a) + B_2 \times G^{-1}(X_m) + B_3 \cdot F^{-1}(X_a) \cdot G^{-1}(X_m) + U,$$

which reduces to

$$Y_p = b_0 + b_1 \cdot X_a + b_2 \cdot X_m + b_3 \cdot (X_a{}^2) + b_4 \cdot (X_m^2) + b_5 \cdot (X_a \cdot X_m) + b_6(X_a{}^2 \cdot X_m) + b_7 \cdot (X_a \cdot X_m^2) + b_8 \cdot (X_a{}^2 \cdot X_m^2) + e. \quad (4)$$

Equation 3 contains only linear and Linear × Linear trend components, and Equation 4 also contains quadratic, Linear × Quadratic, and Quadratic × Quadratic trend components (cf. Cohen & Cohen, 1975).

Birnbaum (1973, 1974) has shown that invalid conclusions can be drawn from multiple regression analyses when a researcher inappropriately assumes that F, G, and H are linear (as in Equation 3) rather than nonlinear (as in Equation 4). The following example illustrates how this can happen when hierarchical regression analysis is used. Suppose that $Y_p = M + U$, but that $G(M) = \sqrt{M}$, so that $M = X_m^2$. Also assume that there is a correlation between X_a and X_m so that $X_m = X_a + V$, where V is the uncorrelated residual variation in X_m that cannot be predicted by X_a. Rewriting the equation for Y_p in terms of the observable measures yields

$$\begin{aligned} Y_p &= M + U = X_m^2 + U, \\ &= X_m \cdot (X_a + V) + U \\ &= (X_m \cdot X_a) + (U + X_m \cdot V) \\ &= (X_m \cdot X_a) + e, \end{aligned}$$

where $e = (U + X_m \cdot V)$ is the correlated residual in the prediction equation. Application of Equation 3 to this model could yield a significant increment in R^2 for the Linear × Linear trend favoring the multiplicative model, despite the fact that performance was actually a simple linear function of the motivation construct. The result might be termed a "spurious" product effect, since the Linear × Linear trend was actually correlated with another unknown factor. This spurious product effect could also be produced by correlations with trend components other than the quadratic trend. For example, the Linear × Linear trend may be correlated with the Quadratic × Quadratic trend.

Theoretically, the problems encountered by monotonic predictors may be overcome by including not only linear and Linear × Linear trend components (as in Equation 3), but also higher order trend components (as in Equation 4). One could test the additive model versus the multiplicative model by comparing the R^2 produced by Equation 4 using unrestricted parameters with the R^2 produced by Equation 4 using the restrictions $b_5 = b_6 = b_7 = b_8 = 0$.

There are practical problems associated with the use of polynomial models such as Equation 4. One is the task of determining the appropriate number of necessary trends (e.g., should cubic trends be included?). The second problem occurs when the predictors contain measurement error. As will be shown later, multiplication of variables containing measurement error greatly amplifies the measurement error problem.

Categories 3 and 4: Monotonic Criterion

In both of these categories $Y_p = H(P)$ is an unknown or partially known monotonic function. As a result, both categories share the common problem that it is theoretically permissable to monotonically transform the criterion measure in order to rescale Y_p in terms of the theoretical construct P. It should be noted that some current measurement theorists (see Krantz & Tversky, 1971) argue that it is only safe to assume a monotonic relation between most "objective" performance measures and the relevant psychological construct. For example, suppose that Y_{p_1} = response latency and Y_{p_2} = response accuracy. Both Y_{p_1} and Y_{p_2} measure an important aspect of the subject's performance. However, it is usually the case that these variables are nonlinearly related to each other (Pachella, 1974), and therefore they cannot both be linearly related to the performance construct.

When $H(P)$ is assumed to be an unknown or partially known monotonic function, it is extremely difficult to interpret the results of hierarchical regression analyses based on the observed measure Y_p. This is because an additive structural model can be transformed into a multiplicative regression model through the monotone function $Y_p = H(P)$. Similarly, a multiplicative structural model can also be transformed into an additive regression model through the function $H(P)$. Thus simply using hierarchical regression analysis to determine whether the Linear × Linear trend is significant is not sufficient for determining whether the underlying structural model is additive or multiplicative.

To show this in a simple manner, assume that $P = A + M$. Next assume that

$$X_a = F(A) = C_1 + \exp(A),$$

$$X_m = G(M) = C_2 + \exp(M),$$

where $\exp(x)$ is the exponential function and the Cs are constants. The original scales can be obtained from the inverse transformations

$$A = F^{-1}(X_a) = \log(X_a - C_1),$$

$$M = G^{-1}(X_m) = \log(X_m - C_2),$$

where $\log(x)$ is the natural logarithm of x. Finally, assume that $Y_p = C_3 + \exp(P) + E_p$, where E_p is the random measurement error associated with Y_p. Then substitution of the above definitions for P, A, and M into the equation for Y_p yields $Y_p = C_3 + (X_a - C_1)\cdot(X_m - C_2) + E_p$, which is a multiplicative regression model similar to Equation 3.

This example shows that it is possible for an additive structural model to be transformed into a multiplicative regression model. The conditions that determine whether or not a transformation to additivity exists provide a possible method for empirically evaluating the structural model. Referring to Equation 3, whenever $b_3 \neq 0$ it will be possible to algebraically rearrange the equation into a pure multiplying model

$$Y_p = [b_0 - (b_1 \cdot b_2)/b_3] + (X_a + b_2/b_3)(b_3 \cdot X_m + b_1) + e,$$

or

$$Y_p = C_1 + (\tilde{X}_a \cdot \tilde{X}_m) + e, \qquad (5)$$

where

$$C_1 = [b_0 - (b_1 \cdot b_2)/b_3],$$

$$\tilde{X}_a = (X_a + b_2/b_3),$$

and

$$\tilde{X}_m = (b_3 \cdot X_m + b_1).$$

Define the joint density function for the pair $(\tilde{X}_a, \tilde{X}_m)$ as $f(\tilde{X}_a, \tilde{X}_m)$ and define the conditional density of $\tilde{X}_a$ given that $\tilde{X}_m = m$ as $f(\tilde{X}_a/m)$. Also define the conditional expected value of Y_p given $\tilde{X}_a = a$ and $\tilde{X}_m = m$ as

$$E[Y_p/a, m] = u(a, m) = C_1 + (a \cdot m),$$

for pairs $(\tilde{X}_a, \tilde{X}_m)$ with $f(\tilde{X}_a, \tilde{X}_m) > 0$. If the product $\tilde{X}_a \cdot \tilde{X}_m$ is always positive, then the monotonic transformation $\log[u(a, m) - C_1] = \log(a) + \log(m)$ produces an additive structural model. Similarly, if $\tilde{X}_a \cdot \tilde{X}_m$ is always negative, then the monotonic transformation $\log[C_1 - u(a, m)] = \log(a) + \log(m)$ also produces an additive structural model.

It is not possible to transform to additivity if a violation of the independence axiom (cf. Krantz & Tversky, 1971) occurs. Assuming that Equation 5 is the correct model, then a violation of the independence axiom occurs when the following two conditions exist: (a) $\tilde{X}_m$ can be fixed at a nonzero value, $\tilde{X}_m = m_1$, and the product $(a \cdot m_1)$ is always positive for all $\tilde{X}_a$ with $f(\tilde{X}_a/m_1) > 0$ in some range $a_1 < \tilde{X}_a < a_2$ and (b) $\tilde{X}_m$ can also be fixed at a nonzero value $\tilde{X}_m = m_2$ and the product $(a \cdot m_2)$ is always negative for all $\tilde{X}_a$ with $f(\tilde{X}_a/m_2) > 0$ in the same interval $a_1 < \tilde{X}_a < a_2$. The first condition states that the slope of the line $u(a, m_1) = [C_1 + (a \cdot m_1)]$ is positive, and the second condition states that the slope of the line $u(a, m_2) = [C_1 + (a \cdot m_2)]$ is negative. This slope reversal is similar to a crossover interaction (cf. Busemeyer, 1980) in analysis of variance.[2]

In practice, it may not be possible to use the test of the independence axiom with the observational method for four reasons. First, violations of independence may occur only for sets of pairs $(\tilde{X}_a \cdot \tilde{X}_m)$ with very low prob-

[2] It is also possible that the slope of the line $(C_1 + am_1)$ is exactly zero for some value of $X_m = m_1$ with $f(X_a/m_1) > 0$ for $a_1 < X_a < a_2$. In this case, it would not be possible to transform the multiplicative model into an additive model.

abilities of occurrence, so that an additive model would be valid for the majority of the values in the range. Second, in order to evaluate the conditional means, $u(a, m)$, it is necessary to know the parameters b_1, b_2, and b_3 in Equation 3. If only small sample estimates are available, then the estimates of b_1 and b_2 are likely to have large standard errors due to multicollinearity associated with the use of product scores.[3] Third, the estimates of b_1, b_2, and b_3 may be severely biased as a result of measurement error, a problem that is described in detail later. Fourth, it is always possible that Equation 3 is the wrong regression model, and for example, Equation 4 is more appropriate. Under certain conditions, Equation 4 can also be factored into a pure multiplicative model given by

$$Y_p = C_0 + (C_1 + C_2 \cdot X_a + C_3 \cdot X_a{}^2) \times (C_4 + C_5 \cdot X_m + C_6 \cdot X_m^2) + e. \quad (6)$$

If the product in Equation 6 always has the same sign, then Equation 6 can be transformed into an additive structural model by a logarithmic transformation. Specifying the regression model as Equation 3 when in fact Equation 4 was the correct model would lead to incorrect estimates of $u(a, m)$ and an invalid test of the independence axiom.[4]

An alternative to ordinary multiple regression analysis is to use a monotone regression procedure (e.g., Young, de Leeuw, & Takane, 1976) that allows the researcher to specify the level of measurement for each variable. There are also limitations associated with this approach. The distribution theory for monotone regression has not been sufficiently developed, so that significance tests for lack of fit are not available. Without rigorous tests for lack of fit, it may be very difficult to reject the additive model (cf. Anderson & Shanteau, 1977).

Summary of the Measurement Level Problem

The previous arguments indicate that when it is theoretically permissable to monotonically transform the criterion variable, then hierarchical regression analysis cannot yield an interpretable test of the multiplicative versus additive structural models. If the criterion is assumed to be a linear measure and the predictors are nonlinear, then it is theoretically possible to use the hierarchical regression method. However, in this case, it is essential that all relevant trend components be included in the model, otherwise spurious results could occur. In practice, it may be very difficult to use the hierarchical regression method with monotonic predictors for two reasons: one is the problem of determining which trend components to include in the model, and the second is the fact that multiplying variables measured with error amplifies the measurement error problem. If one can establish that the criterion and the predictors are linear measures, then a stronger argument can be made for the use of the observational method and hierarchical regression analysis to empirically evaluate the multiplicative structural model. However, even in this case the results may be invalid due to the problem of measurement error (see below).

Measurement Error

In the discussion that follows, the classic true score measurement model (cf. Lord & Novick, 1968) is assumed, so that

$$X_a = A + E_a, \quad (7a)$$

$$X_m = M + E_m, \quad (7b)$$

$$Y_p = P + E_p, \quad (7c)$$

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[3] When the means of X_a and X_m are substantially different from zero, the product score $(X_a \cdot X_m)$ tends to be highly correlated with X_a and X_m. This tends to increase the standard errors of the estimated beta weights for X_a and X_m. Using deviation scores will tend to reduce this problem.

[4] Scheffé (1959, p. 95) discussed a general method for determining whether a two-variable polynomial regression model can be transformed into an additive model when the model parameters are known. Suppose y is the polynomial function of X_a and X_m given by Equation 4. Scheffé has shown that if

$$\left(\frac{\partial^2 y}{\partial X_a \partial X_m}\right) \Big/ \left(\frac{\partial y}{\partial X_a}\right)\left(\frac{\partial y}{\partial X_m}\right) = w(y),$$

where $w(y)$ is an integrable function of y, then a transformation to additivity exists. There are three problems concerning the use of Scheffé's method: (a) the parameters are never known exactly, (b) it may be difficult to prove that no solution $w(y)$ to the functional equation exists, and (c) the solution $w(y)$ may not be integrable.

where the random errors E_a, E_m, E_p are uncorrelated with each other and with the latent random variables A, M, P, and U. The errors have zero means, and the variances are Θ_a, Θ_m, and Θ_p for E_a, E_m, and E_p, respectively. The reliabilities of X_a and X_m can be defined as $\rho_a = V(A)/V(X_a)$ and $\rho_m = V(M)/V(X_m)$, where $V(X)$ is the variance of X.

A serious limitation of the hierarchical regression method for testing additive and multiplicative structural models is the fact that this method does not take into consideration the consequences produced by multiplying predictor variables measured with error (see Arnold, 1982). Ordinary least squares multiple regression analyses assume that the predictor variables are error free. Contrary to Arnold and Evans' (1979) earlier conclusions concerning the appropriateness of the fixed regression model for theory testing, it will be shown here that violations of the assumptions concerning measurement error can lead to faulty inferences concerning the structural parameters. These inferential problems cannot be corrected by increasing sample size or by averaging across replications of research.

In order to reveal the influence of measurement error on the parameters estimated by the hierarchical multiple regression method, it is useful to write the structural equation in terms of the observed measures, which yields

$$\begin{aligned}(Y_p - E_p) &= B_0 + B_1 \cdot (X_a - E_a) + B_2 \cdot (X_m - E_m) \\ &\quad + B_3 \cdot (X_a - E_a) \cdot (X_m - E_m) + U,\end{aligned}$$

or

$$\begin{aligned}Y_p = b_0 + b_1 \cdot X_a + b_2 \cdot X_m \\ + b_3 \cdot (X_a \cdot X_m) + e, \quad (8)\end{aligned}$$

where

$$\begin{aligned}e = U - [B_1 \cdot E_a + B_2 \cdot E_m \\ + B_3 \cdot (X_m \cdot E_a + X_a \cdot E_m - E_a \cdot E_m)] + E_p.\end{aligned}$$

Since the error in Equation 8 is correlated with each of the predictors, X_a, X_m, and $X_a \cdot X_m$, the least squares estimates of $b' = [b_0, b_1, b_2, b_3]$ will be biased and inconsistent estimates of the structural parameters $B' = [B_0, B_1, B_2, B_3]$ from Equation 2 (see Johnston, 1972, chap. 9). The bias from measurement error can severely influence the relative magnitudes and even the signs of the regression coefficients (see Bohrnstedt & Carter, 1971, for examples).

Implications for Multiplicative Structural Models

Define R^2_{mult} as the percentage of variance predicted by Equation 8 with unrestricted parameters, and define R^2_{add} as the percentage of variance predicted by Equation 8 with the restriction that $b_3 = 0$. The increment in R^2, defined as $\Delta R^2 = (R^2_{\text{mult}} - R^2_{\text{add}})$, will be severely attenuated by even moderate measurement error associated with the predictors. To demonstrate this, it is necessary to investigate the asymptotic distribution of ΔR^2 as the sample size N increases to the limit.

Specifically, the symbol $\Delta\rho^2$ will be used to represent the probability limit of ΔR^2 (i.e., ΔR^2 converges in probability to $\Delta\rho^2$ as $N \to \infty$). In the Appendix, the value of $\Delta\rho^2$ is derived based on the following assumptions: (a) X_a, X_m, and Y_p are generated by Equations 7a through 7c, (b) the errors are normally distributed, and (c) the latent variables A and M are distributed multivariate normal with a correlation of $\rho_{A,M}$. (It is not assumed that product $A \cdot M$ will be normally distributed). For simplicity, all observed measures are expressed in deviation score form.

The value of $\Delta\rho^2$ will depend directly on the reliability of the product term[5] $(X_a \cdot X_m)$,

[5] Arnold (1982) has reported a derivation of $\Delta\rho^2$ for the special case where one of the predictors (say X_m) is dichotomous and measured without error (i.e., $\rho_m = 1.0$). However, Arnold's derivation contains an apparent error. Arnold reported that $\Delta\rho^2 = \rho_a{}^2$ [actual effect size]. Actual effect size $= B_3{}^2[1 - R_{(X_aX_m)\cdot X_a,X_m}{}^2]$, and $R_{(X_aX_m)\cdot X_a,X_m}{}^2$ equals the percentage of variance in the product $(X_a \cdot X_m)$ predicted by X_a and X_m. This cannot be correct because $R_{(X_aX_m)\cdot X_a,X_m}{}^2$ will vary depending on the intercepts of the scales X_a and X_m, while $\Delta\rho^2$ is invariant to changes in these intercepts. Also Arnolds's definition allows "actual effect size" to be greater than 1.0 under some circumstances, but $0 < \Delta\rho^2 < 1.0$.

If it is assumed that X_m is dichotomous, $\rho_m = 1.0$, $\rho_{A,M} = 0.0$, and $V(X_a)$ is constant across both levels of X_m, then we can derive the following simple expression for this special case:

which is defined as $\rho_{am} = V(A \cdot M)/V(X_a \cdot X_m)$. The Appendix shows that

$$\Delta\rho^2 = \rho_{am}[B_3{}^2 \cdot V(A \cdot M)]/V(Y_p), \quad (9)$$

where B_3 is the structural parameter associated with the product $(A \cdot M)$ in Equation 2. Bohrnstedt and Marwell (1978) have expressed the reliability of a product (i.e., ρ_{am}), in terms of the reliability of X_a (i.e., ρ_a) and the reliability of X_m (i.e., ρ_m). Under the assumptions stated above (including the use of deviation scores)

$$\rho_{am} = [(\rho_a \cdot \rho_m) + \rho_{A,M}{}^2]/[1 + \rho_{A,M}{}^2], \quad (10)$$

where $\rho_{A,M}$ is the correlation between the latent variables A and M. If X_a and X_m were perfectly reliable, then $\rho_{am} = 1$ and $\Delta\rho^2 = [B_3{}^2 \cdot V(A \cdot M)]/V(Y_p)$. Therefore, the percentage of attenuation in $\Delta\rho^2$ resulting from measurement error is indicated directly by ρ_{am}. When $\rho_{A,M} = 0$, then $\rho_{am} = \rho_a \cdot \rho_m$, so that if one measure were highly reliable (e.g., $\rho_a = .8$) and the other very unreliable (e.g., $\rho_m = .2$), then the reliability of the product would be below the reliability of the less reliable measure (e.g., $\rho_{am} = (.2)(.8) = .16$). Table 2 provides some examples of the reliability of a product (and thus the percentage attenuation) for some sample values of ρ_a, ρ_m, and $\rho_{A,M}$. As can be seen, the value of ρ_{am} is a decreasing function of ρ_a and ρ_m, usually falling below the minimum for (ρ_a, ρ_m). Also, the value of ρ_{am} is an increasing function of $\rho_{A,M}$, so that the attenuation is maximal when $\rho_{A,M} = 0$ (e.g., when the causal variables are orthogonal).

The point of this analysis is that ΔR^2 can be reduced to practically zero if either ρ_a or ρ_m or both are small. The problem of measurement error becomes even more severe when product scores involving more than two fallible measures such as $(X_1 \cdot X_2 \cdot X_3) = (X_1 + E_1)(X_2 + E_2)(X_3 + E_3)$ are used. For example, if it is assumed that X_1, X_2, and X_3 are normally distributed and uncorrelated deviation scores, then the reliability of the three-way product will be the product of the individual reliabilities! This fact highlights the necessity for researchers to recognize the severe attenuation in statistical power to detect Linear × Linear or higher order trends when the predictor variables are measured with error.

Table 2
Effects of the Reliability of X_a (ρ_a), the Reliability of X_m (ρ_m), and the Correlation, $\rho_{A,M}$, Between A and M on the Reliability of the Product ($X_a \cdot X_m$)

ρ_a	ρ_m	$\rho_{A,M}$	ρ_{am}
.2	.2	.00	.04
.2	.2	.15	.06
.2	.2	.32	.13
.2	.8	.00	.16
.2	.8	.20	.19
.2	.8	.40	.28
.5	.5	.00	.25
.5	.5	.25	.29
.5	.5	.50	.40
.5	.8	.00	.40
.5	.8	.30	.45
.5	.8	.55	.54
.8	.8	.00	.64
.8	.8	.30	.67
.8	.8	.63	.74

Note. These values were generated by setting $V(A) = V(M) = 1$. The resulting values of ρ_{am} were determined as a function of $\rho_{A,M}$, ρ_a, and ρ_m. The correlations $\rho_{A,M}$ span the range permitted by the constraints resulting from the reliabilities. These results are based on the use of deviation scores for X_a and X_m.

Errors-in-Variables Regression Models

Many of the problems produced by measurement error may be overcome by using latent structural equation models (e.g., Bentler, 1980; Rock et al., 1977), errors-in-variables regression models (e.g., Warren, White, & Fuller, 1974), or regression models with instrumental variables (Johnston, 1972; James & Singh, 1978). Except for McDonald (1967), few researchers have considered multiplicative latent structural equation models, and there are a number of statistical problems that must be overcome before latent structural equation analyses can be safely used to test multiplicative models.

The latent structural equation approach requires the researcher to obtain estimates of the following two covariance matrices: One

$$\Delta\rho^2 = \rho_a[B_3{}^2 V(A \cdot M)/\sigma_y{}^2]$$

$$= \rho_a[\text{actual effect size}],$$

where actual effect size equals the value of $\Delta\rho^2$ when $\rho_m = \rho_a = 1.0$.

is the covariance matrix for the latent causal variables (A, M, and $A \cdot M$), which will be symbolized as $\mathbf{\Phi}$; the second is the covariance matrix for the measurement errors, symbolized as $\mathbf{\Theta}$, where $\mathbf{\Theta}$ is usually a diagonal matrix. The standard procedure is to use multiple indicators of each construct. Using a congeneric measurement model (cf. Rock et al., 1977), it is possible to obtain consistent estimates of $\mathbf{\Phi}$ and $\mathbf{\Theta}$. Separating true score variance from measurement error variance makes it possible to obtain consistent estimates of the structural parameters. The major problem that is introduced by the multiplicative latent variable ($A \cdot M$) is that estimates of the measurement error variance associated with the multiplicative term $X_a \cdot X_m$ are not readily available. Two different approaches have been suggested to estimate the measurement error variance for the multiplicative term $X_a \cdot X_m$.

One obvious approach to the problem is to obtain several product scores $X_{ai} \cdot X_{mj}$, each being a separate indicator of $A \cdot M$. The measurement error for each product score will be equal to $E_{am} = (A \cdot E_m + M \cdot E_a + E_a \cdot E_m)$. Once multiple indicators of $A \cdot M$ have been constructed, they are introduced into the congeneric measurement model in the same fashion as the indicators for A and M. A recent example of the use of this approach is the analysis of the Fishbein and Ajzen model by Bagozzi (1981). The major problem with this approach is that the statistical properties of the estimates produced by the inclusion of the complex error structure E_{am} are completely unknown. For example, it is not known whether the estimates obtained from this method will be consistent estimates of the structural parameters. Thus, it is risky at best to use the product-score indicator method.

A second approach, suggested by Bohrnstedt and Marwell (1978), depends on prior knowledge of the reliabilities of X_a and X_m. Suppose that the measurement error variances Θ_a and Θ_m associated with X_a and X_m are given either by theory or extensive confirmatory factor analytic research. Then using the formulas provided by Bohrnstedt and Marwell (1978, p. 266) for the variance of a product, it is possible to derive the variance of E_{am} (symbolized as Θ_{am}). If the assumptions described earlier for Equation 9 hold (including the use of deviation scores), then

$$\Theta_{am} = \rho_a \cdot V(X_a) \cdot (\Theta_m) + \rho_m \cdot V(X_m) \cdot (\Theta_a) + (\Theta_a) \cdot (\Theta_m). \quad (11)$$

Having determined the covariance matrix $\mathbf{\Theta} = \mathrm{diag}(\Theta_a, \Theta_m, \Theta_{am})$ for all three variables X_a, X_m, and $X_a \cdot X_m$, it is possible to obtain consistent estimates of the structural parameters by the following modified least-squares method: Define $\mathbf{X}$ as a ($N \times 3$) matrix containing N rows of observations on X_a, X_m, and $X_a \cdot X_m$. Also define $\hat{\mathbf{\Sigma}}_x$ as the unbiased estimate of the covariance matrix for $\mathbf{X}$. Let $\mathbf{y}$ be a column vector containing N observations of the criterion variable measured in deviation score form. Finally, define $\hat{\mathbf{\Sigma}}_{xy}$ as the unbiased estimate of the covariance matrix between y and $\mathbf{X}$. Then a modified least squares estimator is given by

$$\hat{\mathbf{B}} = (\mathbf{\Sigma}_x - \mathbf{\Theta})^{-1}\hat{\mathbf{\Sigma}}_{xy}. \quad (12)$$

Warren et al. (1974) have described this approach in more detail. Alternatively, one could use the latent structural equation model described by Rock et al. (1977), treating $\mathbf{\Theta}$ as a fixed parameter matrix. In sum, the Bohrnstedt and Marwell approach attempts to overcome some of the problems of the product score indicator approach by deriving the measurement error variance of the product scores analytically rather than by estimating it. An example illustrating the use of the Bohrnstedt and Marwell approach is a study by Heise and Smith-Lovin (1981) that investigated a multiplicative impression formation model.

There are two problems with the Bohrnstedt and Marwell approach. First of all, it requires prior knowledge of the reliabilities of X_a and X_m. The second problem is that the derivation of the reliability of a product depends on assumptions of multivariate normality and uncorrelated measurement errors. If the population values of $\mathbf{\Theta}$ are not available, but consistent estimates are available, then Equation 12 will still produce consistent estimates of the structural parameters. However, using sample estimates for $\mathbf{\Theta}$ will also increase the variance of the estimates of **B**. In addition, subtracting a sample value of $\mathbf{\Theta}$ from $\hat{\Sigma}_x$ may produce a singular matrix. (See Warren et al., 1974, for more details on both of these issues). Overall, the Bohrnstedt and Marwell approach appears to be superior to the product score indicator method described earlier.

Summary of Measurement Error Problems

Hierarchical regression analyses will tend to severely underestimate the Linear × Linear trend and higher order trends of the structural model when the predictor variables were measured with error. The increment in R^2 from the product term is directly related to the reliability of the product. Under reasonable conditions, the reliability of the product of several measures is directly related to the product of the reliabilities of the individual measures being multiplied. Thus, the presence of measurement error in the predictor variables will drastically reduce the power to detect a significant contribution from the product term. If the reliabilities of the individual predictor variables are known, then under certain assumptions an errors-in-variables regression procedure suggested by Bohrnstedt and Marwell (1978) will provide consistent estimates of the structural parameters for a multiplicative model.

General Conclusions

If the researcher is interested in empirically evaluating a theory that proposes a multiplicative combination rule, then hierarchical regression analysis used in conjunction with the observational method is inadequate for two reasons: (a) a significant Linear × Linear trend or higher order trends may be eliminated by a theoretically permissable monotonic transformation, and (b) a nonsignificant Linear × Linear trend or higher order trends may result from a drastic reduction in power caused by the multiplication of variables containing measurement error. Simultaneous consideration of the measurement level and measurement error problems makes the results of hierarchical regression analysis extremely difficult to interpret. The measurement level problem implies an increased rate of false detections of multiplicative effects, and the measurement error problem implies a decreased rate of detection of true structural multiplicative effects. Unfortunately, one cannot easily identify the form of the associated operating characteristic curve, which, in turn, makes it difficult to identify the optimal decision-making strategy. A test of the independence axiom and an errors-in-variables regression procedure were proposed as possible ways of eliminating these two problems. However, the correct use of those procedures requires precise estimates of the measurement error variances and the regression coefficients.

Alternatives to the observational method are the use of experimental designs, conjoint measurement methods (Krantz & Tversky, 1971), functional measurement methods (Anderson, 1982, chap. 5), and "scale free" or "scale convergence" methods (Birnbaum, 1982).

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Appendix

The purpose of this appendix is to determine the magnitude of the asymptotic bias in the estimate of the increment in R^2 for the Linear $\times$ Linear trend estimated by hierarchical regression when the predictor variables are measured with error. The column vector $\mathbf{Y}$ represents N observations obtained from Y_p, and the $(N \times 3)$ matrix $\mathbf{X}$ represents N observations on X_a, X_m, and $(X_a \cdot X_m)$. The column vector $\mathbf{y}$ represents the deviation score measures of Y, and $\mathbf{x}$ represents the deviation score measures of $\mathbf{X}$. The development below is based on these deviation scores.

It is assumed that Y_p, X_a, and X_m were generated by Equations 7a–7c. In addition, the errors E_a and E_m are assumed to be independently normally distributed. Finally, it is assumed that the latent random variables A and M are distributed according to a multivariate normal distribution.

On the basis of Equations 7a and 7c, the matrix of predictor deviation scores can be decomposed into $\mathbf{x} = \mathbf{Z} + \mathbf{V}$, where $\mathbf{Z}$ is the matrix of true scores and $\mathbf{V}$ is the matrix of measurement errors corresponding to each value in $\mathbf{x}$. The deviation score form of the structural model can then be written as

$$\mathbf{y} = \mathbf{ZB} + \mathbf{U} = (\mathbf{x} - \mathbf{V})\mathbf{B} + \mathbf{U} = \mathbf{xB} + (\mathbf{U} - \mathbf{VB}) = \mathbf{xB} + \mathbf{e}, \tag{A1}$$

where $\mathbf{B}' = [B_1, B_2, B_3]$ is the vector of structural parameters, and $\mathbf{e}$ is a vector of residuals that are correlated with $\mathbf{x}$.

The covariance matrix for the latent variables $[A, M, \text{and } (A \cdot M)]$ is given by

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & & \\ \phi_{21} & \phi_{22} & \\ 0 & 0 & \phi_{33} \end{bmatrix}$$

where $\phi_{11} = V(A)$, $\phi_{22} = V(M)$, $\phi_{21} = \mathrm{COV}(A, M)$, and $\phi_{33} = V(A) \cdot V(M) + \mathrm{COV}(A, M)^2$ (see Bohrnstedt and Goldberger, 1969). The error for the multiplicative term will equal $E_{am} = [A \cdot E_m + M \cdot E_a + E_a \cdot E_m]$. The covariance for the errors $[E_a, E_m, E_{am}]$ is given by

$$\mathbf{\Theta} = \begin{bmatrix} \Theta_{11} & & \\ & \Theta_{22} & \\ & & \Theta_{33} \end{bmatrix}$$

where $\Theta_{11} = V(E_a)$, $\Theta_{22} = V(E_m)$, and $\Theta_{33} = V(E_{am}) = V(A) \cdot V(E_m) + V(M) \cdot V(E_a) + V(E_a) \cdot V(E_m)$ (see Bohrnstedt & Goldberger, 1969).

The covariance matrix for the observed measures in $\mathbf{x}$ will be symbolized as $\mathbf{\Sigma}_x$ with elements σ_{ij}, and is given by

$$\mathbf{\Sigma}_x = \mathbf{\Phi} + \mathbf{\Theta}.$$

The ordinary least squares estimator for $\mathbf{B}$ is obtained from the following equation:

$$\mathbf{b} = (\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{y}. \tag{A2}$$

To find the asymptotic value of R^2 it is necessary to obtain the probability limit of the ordinary least squares estimator, symbolized as plim($\mathbf{b}$). Using the derivation by Johnston (1972, p. 282),

$$\operatorname{plim}(\mathbf{b}) = [\mathbf{I} - (\mathbf{\Sigma}_x{}^{-1}\mathbf{\Theta})]\mathbf{B}, \tag{A3}$$

or

$$\operatorname{plim}(b_3) = (\phi_3/\sigma_{33})B_3,$$

where b_3 is the regression coefficient associated with the product term and σ_{33} is the variance of the product.

The second step is to find the probability limit for the R^2 predicted by Equation A1, symbolized as $\operatorname{plim}(R^2_{\text{mult}})$. One useful formula for R^2 is the following:

$$R^2_{\text{mult}} = \frac{(\mathbf{b}'\hat{\mathbf{\Sigma}}_x\mathbf{b})}{\hat{\sigma}_y{}^2}. \tag{A4}$$

Taking the probability limit yields

$$\operatorname{plim}(R^2_{\text{mult}}) = [\operatorname{plim}(\mathbf{b})]'\mathbf{\Sigma}_x[\operatorname{plim}(\mathbf{b})]/\sigma_y{}^2. \tag{A5}$$

Using a similar argument, one can obtain the probability limit for the R^2 predicted by the additive model obtained by setting $B_3 = 0$ in Equation A1. The probability limit for the R^2 predicted by the additive model will be symbolized as $\text{plim}(R^2_{\text{add}})$. The increment in R^2 resulting from the Linear × Linear trend is obtained from $\Delta R^2 = (R^2_{\text{mult}} - R^2_{\text{add}})$. The probability limit for ΔR^2 will equal

$$\text{plim}(\Delta R^2) = \Delta\rho^2 = \rho_{am}[B_3{}^2\phi_{33}]/\sigma_y{}^2, \tag{A6}$$

where $\Delta\rho^2$ is a symbol representing plim (ΔR^2). In Equation A6, $\rho_{am} = (\phi_{33}/\sigma_{33})$.

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