Interference effects of choice on confidence: Quantum characteristics of evidence accumulation

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Decision-making relies on a process of evidence accumulation which generates support for possible hypotheses. Models of this process derived from classical stochastic theories assume that information accumulates by moving across definite levels of evidence, carving out a single trajectory across these levels over time. In contrast, quantum decision models assume that evidence develops over time in a superposition state analogous to a wavelike pattern and that judgments and decisions are constructed by a measurement process by which a definite state of evidence is created from this indefinite state. This constructive process implies that interference effects should arise when multiple responses (measurements) are elicited over time. We report such an interference effect during a motion direction discrimination task. Decisions during the task interfered with subsequent confidence judgments, resulting in less extreme and more accurate judgments than when no decision was elicited. These results provide qualitative and quantitative support for a quantum random walk model of evidence accumulation over the popular Markov random walk model. We discuss the cognitive and neural implications of modeling evidence accumulation as a quantum dynamic system.

confidence | Markov | decision-making | cognitive model | random walk

Decisions in a wide range of tasks (e.g., inferring the presence or absence of a disease, the guilt or innocence of a suspect, and the left or right direction of enemy movement) require evidence to be accumulated in support of different hypotheses. Arguably, the most successful theory of evidence accumulation in humans and other animals is Markov random walk (MRW) theory (and diffusion models, their continuous space extensions) (1, 2). MRWs can be viewed as psychological implementations of a first-order Bayesian inference process that assigns a posterior probability to each hypothesis (3). MRWs can account for choices, response times, and confidence for a variety of different decision types (2, 4). Moreover, these models of the accumulation process have been connected to neural activity during decision-making (5, 6).

According to MRW models, when deciding between two hypotheses, the cumulative evidence for or against each hypothesis realizes different levels at different times to generate a single particle-like trajectory of evidence levels across time (Fig. 1). At any point in time, the decision-maker has a definite level of evidence, and choices are made by comparing the existing level of evidence against a criterion. Evidence above the criterion favors one option, and evidence below it favors the alternative. Other responses are modeled in a similar manner; for example, confidence ratings are modeled by mapping evidence states onto one or more ratings (4). However, this idea that judgments and decisions are simply read out from the existing level of evidence—henceforth referred to as the "read-out" assumption—is inconsistent with the well-established idea that preferences and beliefs are constructed rather than revealed by judgments and decisions (7).

We present an alternative model of choice and judgment based on quantum random walk (QRW) theory (8–11), which posits that preferences and beliefs are constructed when a judgment or decision is made. Note that this work does not make the assumption that the brain is a quantum computer; instead, we simply use the mathematics of quantum theory to explain and predict human behavior. According to QRW theory, at any point in time before a decision, the decision-maker is in a superposition state that is not located at a single level of evidence. Instead, each level of evidence has a potential to be expressed, formalized as a probability amplitude (Fig. 1). New information changes the amplitudes, producing a wavelike process that moves the amplitude distribution across time.

In some ways the QRW is like a second-order Bayesian model (12). According to the latter, the decision-maker assigns a probability (rather than an amplitude) to each level of evidence for each hypothesis. However, like the MRW model, second-order Bayesian models are perfectly compatible with the read-out assumption, and as an optimal model, this would suggest that a decision should not change the probability assigned to each evidence level. In contrast, a QRW, like all quantum models of cognition (13), treats a judgment or decision as a measurement process that constructs a definite state from an indefinite (superposition) state. When a decision is made, the indefinite state collapses onto a set of evidence levels that correspond to the observed choice, producing a definite choice state. Confidence ratings work similarly, with the indefinite state collapsing onto a more specific set of levels corresponding to the observed rating.

These different theories of choice and judgment have strong implications for sequences of responses. Consider the situation when decision-makers have to make a choice (e.g., decide that hypothesis A or B is true) and later rate their confidence that a given (usually the chosen) hypothesis is true. According to the read-out assumption, a choice is reported on the basis of existing evidence that does not change the internal state of evidence itself. This applies to the MRW, a second-order Bayesian model, and many other accumulation models as well. Thus, after pooling across a person's choices, the distribution of confidence ratings should be identical to conditions in which the person makes no choice at all. By contrast, the state of the system in a QRW

Significance

Most cognitive and neural decision-making models—owing to their roots in classical probability theory—assume that decisions are read out of a definite state of accumulated evidence. This assumption contradicts the view held by many behavioral scientists that decisions construct rather than reveal beliefs and preferences. We present a quantum random walk model of decision-making that treats judgments and decisions as a constructive measurement process, and we report the results of an experiment showing that making a decision changes subsequent distributions of confidence relative to when no decision is made. This finding provides strong empirical support for a parameter-free prediction of the quantum model.

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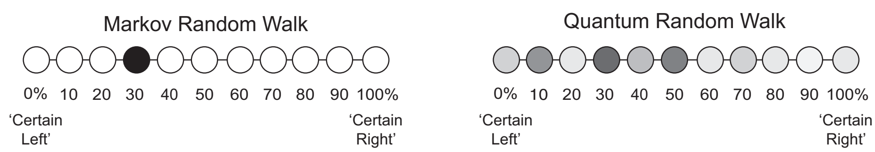

Fig. 1. Diagram of a state representation of a Markov and a quantum random walk model. In the Markov model, evidence (shaded state) evolves over time by moving from state to state, occupying one definite evidence level at any given time. In the quantum model the decision-maker is in an indefinite evidence state, with each evidence level having a probability amplitude (shadings) at each point in time.

is changed when a choice creates a definite state. Subsequent processing starts from the definite state, and the amplitudes spread out again. Thus, if information processing continues after the initial stage, the QRW predicts an interference effect where the marginal distribution of confidence judgments following a choice will differ from a condition in which no choice is made.

A proof of the predicted interference effect for QRWs is in *SI Appendix*. The proof shows that the interference effect of choice on confidence is the result of the interaction between the creation of a definite state and subsequent evidence accumulation after making a choice. Subsequent or second-stage processing is a necessary condition for the effect. Critically, second-stage processing occurs when people are asked to report a confidence rating following a choice, giving rise to response reversals (14) and other properties (15). We also provide a proof that MRWs predict no difference between the marginal distributions of confidence ratings (i.e., no interference) regardless of the presence of second-stage processing. This proof holds for a large range of MRWs, including ones with decay (16), leakage of evidence (17), and trial-by-trial variability in the decision process (18).

Empirical Test of Predicted Interference Effect

We tested these opposing predictions concerning interference effects using a perceptual task that requires participants to judge the direction of motion in a dynamic dot display (Fig. 2). Specifically, nine participants completed 112 blocks of 24 trials each over five 1-h experimental sessions, a total of 2,688 trials per person (*SI Appendix*). During each trial, participants viewed a random dot motion stimulus that consisted of moving white dots in a circular aperture on a black background (19). A percentage of the dots moved coherently in one direction (left or right), and the rest moved randomly. Difficulty was manipulated between trials by changing the percentage of coherently moving dots (2%, 4%, 8%, or 16%). In the choice condition—half of the randomly ordered blocks—participants were prompted 0.5 s from stimulus onset via a low-frequency beep (400 Hz) to decide whether the coherently moving dots were moving left or right and entered their choice by clicking the corresponding mouse button. In the no-choice condition—the other half of the blocks—participants were prompted 0.5 s from stimulus onset via a high-frequency beep (800 Hz) to make a motor response (click the left or right mouse button as instructed). In all trials, the stimulus remained on screen for a second stage of processing after the choice or click. After an additional 0.05, 0.75, or 1.5 s following the first response, participants were prompted via a second beep (400 Hz) to rate their confidence that the coherently moving dots were moving right on a semicircular scale that appeared at the time of the prompt, ranging from 0 (certain left) to 100% (certain right) in unit steps. Note that to match the overall processing time of the stimulus across conditions, the confidence prompt was time-locked to the initial choice or click entry.

For the behavioral analyses, we collapsed confidence responses across the dot motion direction, recoding confidence onto a half scale (50% guess to 100% certain). All behavioral analyses were conducted using hierarchical Bayesian general linear models (20). The coefficient b is the linear effect of a predictor on the criterion. We also report the highest density interval (HDI) for all estimates, which specifies the range covering the 95% most credible values of the posterior estimates. A normal link was used for confidence judgments after transforming them to log odds, and a logistic link was used for choices.

On average, confidence increased with motion coherence ($b = 0.66$; 95% HDI = [0.31, 1.02]). In the choice condition, the proportion of correct choices increased with coherence ($b = 0.50$; 95% HDI = [0.04, 1.16]). Confidence judgments were, on average, lower in the choice ($M = 83.96$; $SD = 15.56$) than in

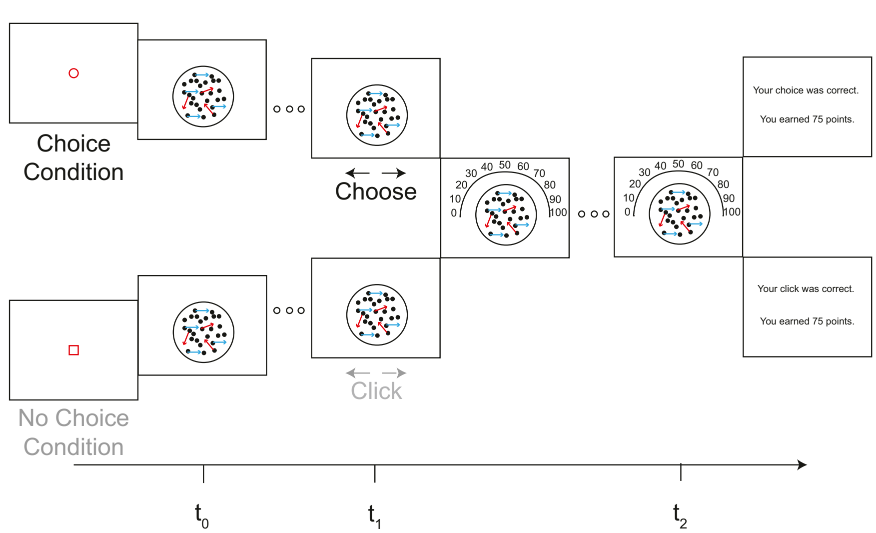

Fig. 2. Diagram of the task. A fixation point indicated the choice/no-choice condition, then the stimulus was shown for 0.5 s. A prompt (t_1) then asked for a decision on the direction of the dot motion (choice condition) or a motor response (no-choice condition). The stimulus remained on the screen. A second prompt (t_2) then asked for a confidence rating on the direction of the dot motion. Finally, feedback was given on the accuracy of their responses.

the no-choice condition ($M = 85.15$; $SD = 14.95$); the value of the interference main effect coefficient is reported at the individual and group level in Table 1 (see also *SI Appendix*, Table S1). *SI Appendix*, Figs. S1 and S6 also show this effect at the distribution level. This effect of choice on confidence provides evidence against the read-out assumption and is consistent with the quantum claim that choice changes the state of the cognitive system.

We also examined the QRW prediction that the interference effect does not occur with choice alone but that the interaction between choice and subsequent processing creates interference. To gauge whether participants were sampling information during the second stage, we examined whether there was an interaction between coherence and duration of the second stage when confidence was predicted on a full scale from 0 (completely sure, incorrect direction) to 100 (completely sure, correct direction). The magnitude of this coefficient indicates second-stage processing under the following logic: on average, people sample information in favor of the correct answer during this second stage, and this evidence should be stronger with higher coherence (drift), resulting in an interaction between coherence and the duration of the second stage.

The value of this second-stage processing interaction coefficient is reported in Table 1. With the exception of participant 3, there is a 1:1 correspondence between credible second-stage processing and a credible interference effect. Note that participant 3 is by several measures an anomaly: this participant was unable to distinguish between dot motion directions in most conditions and in fact had lower than 50% correct choices in some conditions (*SI Appendix*, Fig. S4).

Further evidence for the requirement of second-stage processing comes from a study in which we failed to obtain interference when no second-stage processing was induced. This study was almost identical to the one described above, with two differences: first, there was no trial-by-trial feedback, and second, the decision time was 0.8 s rather than 0.5 s. The result of these differences is that participants did not pay attention to the stimulus after giving their initial choice or click response. This is evidenced by the lack of credible second-stage processing in this experiment (*SI Appendix*).

Finally, we examined how the accuracy of the confidence ratings changed as a result of this interference. To this end, we coded whether the confidence rating fell on the correct side of the scale relative to the actual left/right motion direction. In the choice and no-choice conditions, confidence ratings were on the correct side of the scale in 76.36% and 76.25% of cases, respectively. The accuracy of the confidence ratings in these conditions was credibly the same when we fit each one with a one-parameter Bernoulli distribution ($M_{\text{diff}} = 0.11\%$; 95% HDI $[-0.04\%, 0.20\%]$). The difference between the average confidence ratings and confidence accuracy is called bias, and it measures how well calibrated confidence ratings are to the proportion of times the target event actually occurs. The average bias statistic was 7.66, implying mild overconfidence (95% HDI $[-2.09, 17.66]$). Focusing on the interference effect, we found that overconfidence was lower, and thus, confidence was better calibrated in the choice condition than in the no-choice condition (8.20 vs. 7.13; $M_{\text{diff}} = 1.07$; 95% HDI $[0.29, 1.85]$) (for a similar result, see also ref. 21).

Direct Comparison of QRW and MRW

The interference effect of choice on subsequent confidence provides empirical support for a QRW theory of evidence accumulation over an MRW theory. However, the question remains if the QRW can provide a parsimonious account of the choice and confidence data. Therefore, we compared the QRW to a matched MRW using Bayesian model comparison methods. Both models used drift, diffusion, starting point variability, and attenuation parameters. Versions of the MRW with these parameters or more restricted ones (e.g., without attenuation) have been shown to account well for choice, response time, and confidence data, so superior performance would indicate that the QRW is a particularly viable model (2, 4).

Below we describe each model in mathematical detail (see also *SI Appendix*, Fig. S3, for a visual walk-through). In line with the established mathematical principles governing both the MRW and QRW theories, each one updates the state following a choice at time t_1 to produce a conditioned state that is consistent with the observed choice—for quantum theory, this is referred to as Lüder's rule. However, the two models differ with respect to the effect of this conditioning on the marginal distribution of confidence ratings. As our proof shows, the Markov model obeys the law of total probability, so that after pooling across choices the effect of conditioning on the choice completely disappears as if no choices were made. This is because the choice does not change the location of the evidence, only the information an external observer has about it. In contrast, the QRW violates the law of total probability because choice interacts with later evidence accumulation, making pooled confidence ratings after choice diverge from the no-choice case. This is because a definite state consistent with the choice is created, and subsequent dynamics are applied to the revised state, thereby producing the interference effect.

MRW. The MRW used $m = 103$ evidence states $x \in \{-1, 0, \ldots 101\}$. This state space allowed us to assume that states $x = 0, 1, \ldots 100$ corresponded directly with the $n = 101$ confidence ratings (0, 1, ... 100%). The two boundary states (−1,101) served as reflecting boundaries that restricted the process to the range of the confidence scale, and these states were mapped onto confidence ratings of 0 and 100%, respectively. Using 103 states produces a Markov process that closely approximates a continuous state process, and increasing the number of states by refining the state space produces practically the same predictions.

According to the MRW, a decision-maker is in exactly one evidence state at any given time. However, the decision-maker's state is unknown to the observer, and a probability distribution across evidence states is therefore used to represent the state. This

Table 1. Summary of model comparison and statistical effects

Participant	Interference*	Second-stage processing†	Log Bayes factor
1	−0.18 [−0.26, −0.11]‡	0.12 [0.08, 0.18]‡	212
2	−0.15 [−0.23, −0.07]‡	0.08 [0.03, 0.14]‡	41
3	−0.15 [−0.22, −0.07]‡	0.01 [−0.04, 0.06]	−131
4	−0.14 [−0.23, −0.07]‡	0.10 [0.04, 0.15]‡	190
5	−0.11 [−0.19, −0.04]‡	0.07 [0.02, 0.13]‡	837
6	−0.08 [−0.16, −0.01]‡	0.13 [0.07, 0.18]‡	223
7	−0.07 [−0.15, 0.01]	−0.01 [−0.07, 0.05]	−148
8	−0.05 [−0.14, 0.02]	0.04 [−0.08, 0.10]	339
9	−0.01 [−0.09, 0.07]	−0.02 [−0.06, 0.04]	150
Group level	−0.11 [−0.18, −0.04]‡	0.06 [0.01, 0.12]‡	1,713

*Mean posterior coefficient and 95% HDI for main effect of the choice manipulation on half-scale confidence.
†Mean posterior coefficient and 95% HDI for the interaction between coherence and second stage processing time on full-scale confidence.
‡The 95% HDI excluded zero.

distribution is defined by a mixed state vector $\phi(t)$ of dimension $m \times 1$, which gives the probability of being in state x at time t,

$$Pr(x|t) = \phi_x(t). \quad \textbf{[1]}$$

The probability distribution $\phi(0)$ specifies the decision-maker's initial state, which is set as a uniform distribution centered on $x = 50$. The width w is a free parameter indexing trial-by-trial variability in the initial state.

As the decision-maker considers information, the process moves from state to state. An $m \times m$ transition matrix **P** specifies the probability that the process moves from one state to another after some period, so that the probability distribution over evidence states after time t is

$$\phi(t) = \mathbf{P}(t) \cdot \phi(0). \quad \textbf{[2]}$$

Choice probability and confidence are determined as follows. Define a response operator M_R, which is a diagonal matrix with 0.5 located in the row for confidence level 50, ones located in rows for confidence levels 51 through 101, and zeros otherwise. The probability of choosing right at time t_1, denoted $p(R|t_1)$, equals the sum of the projection $\mathbf{M}_R \cdot \phi(t_1)$. The probability of choosing left at time t_1 is $1 - p(R|t_1)$. If right motion is chosen, then this provides information on the location of the evidence (e.g., evidence is at or above state 50), and the probability distribution over the states is updated to $\phi(t_1|R) = \frac{\mathbf{M}_R \cdot \phi(t_1)}{p(R|t_1)}$. Note that if a person were to choose left motion, the response operator $\mathbf{M}_R$ would be replaced by $\mathbf{M}_L$; the two are identical except that the 1 and 0 entries along the main diagonal are flipped.

For confidence ratings, define $\mathbf{M}_y$ as a diagonal matrix with 1 located in the row(s) corresponding to confidence y and zeros otherwise. In the choice condition, the probability of choosing confidence level y at time t_2 following a right motion choice then equals the sum of the projection $\mathbf{M}_y \cdot \mathbf{P}(t_2) \cdot \phi(t_1|R)$. In the no-choice condition, the probability of choosing confidence level y at time t_2 equals the sum of the projection $\mathbf{M}_y \cdot \phi(t_2)$.

The transition matrix is constructed from an $m \times m$ intensity matrix **Q** using the Kolmogorov forward equation so that

$$\mathbf{P}(t) = \exp(\mathbf{Q}t\gamma), \quad \textbf{[3]}$$

where exp is the matrix exponential function and γ is a parameter describing the proportion of time spent processing information up to time t. Consistent with recent work in modeling postdecisional processing (15), this was set to $\gamma = 1$ during the first stage of processing (t_0 to t_1) but was free to vary during the second stage to account for attenuation in incoming information following the first response.

The entries $q_{j,k}$ of the intensity matrix are

$$q_{j,j} = -\sigma^2, \quad \textbf{[4a]}$$

$$q_{j-1,j} = \frac{1}{2}\left(\sigma^2 - \delta\right), \quad \textbf{[4b]}$$

$$q_{j+1,j} = \frac{1}{2}\left(\sigma^2 + \delta\right). \quad \textbf{[4c]}$$

This definition of the intensity matrix was chosen so that the discrete state Markov process closely approximates a continuous state Wiener diffusion process (8). The drift rate δ determines the probability that the process steps toward the true dot motion direction. We scaled the drift rate directly from the percentage of coherently c moving dots so that

$$\delta = \mu \cdot c. \quad \textbf{[5]}$$

If the dots are moving left, c is negative. The parameter μ is a free parameter indexing sensitivity to the coherence. The parameter σ^2 is a diffusion rate controlling the dispersion of the process. This MRW operates on a finite state space, so we set the states $x = -1$ and $x = 101$ as reflecting boundaries to allow the process to continue its evolution after it reaches the finite limits, $-q_{1,1} = q_{1,2} = \sigma^2$ and $q_{102,103} = -q_{103,103} = \sigma^2$.

QRW. The QRW also used $m = 103$ evidence states as in the MRW, similarly assuming that states $x = 0, 1, \ldots 100$ corresponded directly with the 101 confidence ratings and that states $x = -1$ and $x = 101$ were reflecting boundaries which mapped onto confidence ratings of 0 and 100%.

According to the QRW, a decision-maker is not necessarily in any one evidence state at any given time. This uncertainty on the part of the decision-maker is modeled with a superposition state vector $\psi(t)$ of size $m \times 1$, which gives the probability amplitude at the xth evidence level at time t. The probability of observing state x at time t is the squared length of the amplitude in the corresponding row:

$$Pr(x|t) = |\psi_x(t)|^2. \quad \textbf{[6]}$$

The state vector $\psi(0)$ specifies the initial superposed evidence state. We set the probability amplitudes across these states to be uniformly and symmetrically distributed around $x = 50$. The width w of this distribution is a free parameter representing initial uncertainty.

As information is processed, the superposition state drifts over time until a response is elicited. The $m \times m$ unitary matrix operator **U** evolves the amplitudes over time, so that

$$\psi(t) = \mathbf{U}(t) \cdot \psi(0). \quad \textbf{[7]}$$

Choice probability and confidence are determined as follows. We define $\mathbf{M}_R$ in a similar manner as in the MRW. It is a diagonal matrix with $\frac{1}{\sqrt{2}}$ in the row for confidence level 50, ones located in rows for confidence levels 51 through 101, and zeros otherwise. The probability of choosing right at time t_1, denoted $Pr(R|t_1)$, equals the squared length of the projection $\mathbf{M}_R \cdot \psi(t_1)$. The probability of choosing left is $1 - Pr(R|t_1)$. If right motion is chosen, the superposition state is projected onto the corresponding evidence levels, and the probability amplitude is updated to $\psi(t_1|R) = \frac{M_R \cdot \psi(t_1)}{\sqrt{Pr(R|t_1)}}$. If left motion is chosen, $\mathbf{M}_R$ is replaced by $\mathbf{M}_L$; the two are identical except that the 1 and 0 entries along the main diagonal are flipped.

Subsequent processing starts from this new state so that in the choice condition the probability of choosing confidence level y at time t_2 after choosing right then equals the squared length of the projection $\mathbf{M}_y \cdot \mathbf{U}(t_2) \cdot \psi(t_1|R)$. In the no-choice condition, no projection is done at t_1, and the probability of choosing confidence level y at time t_2 equals the squared length of the projection $\mathbf{M}_y \cdot \psi(t_2)$.

The unitary matrix is constructed from a Hamiltonian matrix **H** using the Schrödinger equation so that

$$\mathbf{U}(t) = \exp(-it\mathbf{H}\gamma). \quad \textbf{[8]}$$

The attenuation parameter γ operates in the same way as in the MRW. The entries $h_{j,k}$ of the Hamiltonian matrix are

$$h_{j,j} = \delta \cdot j/m, \quad \textbf{[9a]}$$

$$h_{j-1,j} = h_{j+1,j} = \sigma^2. \quad \textbf{[9b]}$$

This definition of the Hamiltonian matrix was chosen so that the discrete state quantum process closely approximates the continuous state Schrödinger process (9). The δ and σ^2 parameters of the QRW have a similar effect as their counterparts in the MRW but function differently. The diffusion coefficient σ^2 controls the rate at which amplitude flows out of the states. The drift rate δ determines the rate at which probability amplitude flows in. The drift rate δ was set to be a multiplicative function of coherence

(Eq. **5**). [Eq. **9** is a linear potential function in the diagonal of the Hamiltonian (multiplying drift by the state index) so there is a constant positive force pushing evidence toward the correct direction. However, other potential functions (e.g., quadratic) should be investigated in the future.]

The interference effect arises because the amplitudes in states 0–50 interact with those in 50–100 in the no-choice condition, pushing each other outward toward more extreme evidence states. This pressure is not present in the choice condition, leading to less extreme evidence and hence confidence ratings. One consequence of these less extreme confidence ratings is less overconfidence in the choice condition.

Model Comparison. Each model has four free parameters: a parameter that sets the drift as a scalar function of motion direction coherence (μ), a diffusion parameter (σ^2), a second-stage attenuation parameter that dampens of the rate of incoming information after making a choice (γ), and a parameter that determines the width of the initial state distribution (starting point variability) (w) (*SI Appendix*, Table S3). Non-decision time parameters, accounting for components of the response time exogenous to the evidence accumulation process, had limited influence on model fits and were dropped in order to facilitate model estimation.

The models were compared at the individual level: for each participant and each model, four parameters were used to account for 2,688 trials across 24 experimental conditions. Despite having the same number of parameters, the QRW may be functionally more complex, allowing it to produce good fits to the data without necessarily bearing any relationship to the underlying process. To account for this, we compared the Bayes factor between the two models for each participant (22).

The Bayes factor was calculated using a fine-grid approximation across all possible combinations of the four parameters to compute the likelihood function and uniform priors over their values. The results are summarized in Table 1; the log Bayes factor indicates the log odds of the QRW model over the MRW given the data (see *SI Appendix*, Tables S4 and S5, for the maximum likelihoods and parameter estimates).

The Bayes factor for seven out of nine participants and the group level factor decisively favored the QRW (maximum likelihoods yield the same conclusion). Participant 7 did not show second-stage processing or an interference effect, so the MRW may well describe the behavior of this participant. Participant 3 was unable to distinguish between dot motion directions in many conditions, which caused difficulty in fitting both models (*SI Appendix*, Fig. S4). Future model development incorporating methods for mapping evidence to confidence (e.g., using only 0/10/20% or 0/50/100% ratings) could potentially improve fits, but this does not affect interference so we favor simpler, more parsimonious models here.

Fig. 3 illustrates the fit of each model to the choice proportions for each coherence condition and the distribution of confidence in the choice and no-choice conditions for one participant and coherence level [all participants across conditions are given in *SI Appendix*, Fig. S4 (see also *SI Appendix*, Figs. S5 and S6)]. There are several reasons that the QRW gives a better account of the data than the MRW. First, the MRW predicts identical marginal distributions of confidence ratings between choice and no-choice conditions, whereas the quantum model picks up the slight rightward shift of these ratings in the no-choice condition; this phenomenon is the interference effect we described (see *SI Appendix*, Fig. S6; the QRW posterior predictions yield a group mean shift in confidence of +0.66%, compared to +1.19% in the data). Second, the QRW was often better able to simultaneously capture choices along with confidence ratings across the various conditions, whereas the MRW often had to sacrifice or compromise between the two. Notably, the MRW underestimated choice proportions because higher diffusion more accurately captured confidence distributions but at the cost of predicting lower choice accuracy. Finally, the observed confidence distributions are frequently multimodal and discontinuous. The MRW again does not account for these properties. By contrast, the QRW accounts for all of these characteristics in a parsimonious way, operating only on its first principles to earn a superior Bayes factor.

Although this MRW and similar versions have been used to model a wide range of choice and judgment data, it may struggle to account for this data simply because it cannot account for the interference effect. To examine this possibility, we tested a second model—the MRW-E—which assumes that additional evidence may have been accumulated in the no-choice condition, producing more extreme confidence ratings and thus an interference effect. Despite the added ability to produce interference, the QRW still outperformed the MRW-E. The Bayes factor for seven out of nine participants and the group level factor again decisively favored the QRW over the MRW-E. In comparison with the MRW, the MRW-E provides a largely equivalent or often poorer fit in terms of Bayes factors (*SI Appendix*, Table S6 and Fig. S7). Part of the reason the MRW-E does poorly, in addition to the characteristics it inherits from the MRW, is that it assumes more evidence is accumulated in the no-choice condition producing a

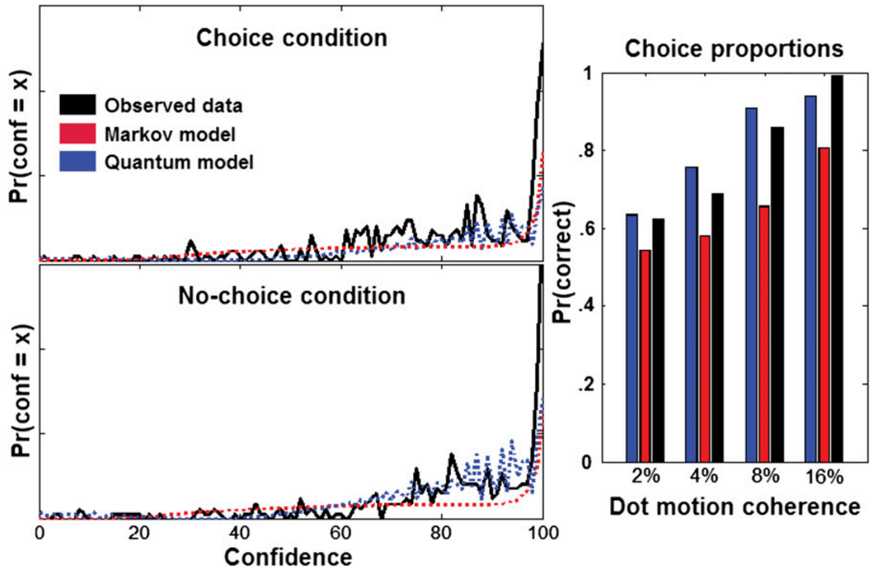

Fig. 3. Data and model fits for participant 4 with coherence level 8%.

change in the accuracy of the confidence ratings as well as the mean shift in confidence. Recall, however, that there is no credible change in the accuracy of the confidence ratings in the data.

Discussion

In this paper, we have developed a model of evidence accumulation during judgment and decision-making based on quantum random walk theory. The QRW represents a point of departure in modeling evidence accumulation from the more typical classical probability approach. In the classical case, evidence evolves over time, but judgments and decisions are simply read out from an existing state without changing the internal state of evidence. In the quantum case, evidence also evolves over time, but judgments and decisions are measurements that create a new definite state from an indefinite (superposition) state. This quantum perspective reconceptualizes how we model uncertainty and formalizes a long-held hypothesis that judgments and decisions create rather than reveal preferences and beliefs. The different approaches make competing a priori predictions for the effect of sequences of responses, and we have shown strong empirical support for the quantum prediction that choices interfere with subsequent confidence judgments. Moreover, we have shown for the first time to our knowledge that the QRW is a viable competitor to the MRW in quantitatively fitting choice and confidence distributions. Note that the QRW can also account for response time distributions (8) and can outperform Markov models in this area as well (10).

A pertinent question is whether the MRW can be adapted to account for the phenomena we observed. This is certainly possible but may prove difficult: as we have shown, our results provide several constraints on potential adaptations. The interference effect itself is a strong constraint: many versions of the MRW that commonly give good accounts of choice and confidence data do not predict any interference.

A second constraint is how the interference effect occurred. In particular, confidence was less extreme following a choice. This poses a problem for explanations like the confirmation bias, where people focus on evidence that justifies their decision after making a choice, meaning they should be more confident in the choice condition (23, 24). Moreover, confidence accuracy also did not change. This poses a problem for models like the MRW-E that assume different amounts of processing between the choice and no-choice conditions.

A third constraint is that the interference effect only occurred when there was second-stage processing. This result poses problems for explanations based on differences in the mapping of evidence onto confidence (25) and explanations assuming that the act of making a choice introduces error into the cognitive system. Both explanations would fail to explain why choice alone (without second-stage processing) does not interfere with confidence. Alternatively, on some trials during the choice condition, participants reversed their initial choice (14) and could have reported unexpectedly low confidence on these trials, producing the interference effect. However, reversals during the choice condition happened infrequently (6.1%), and confidence on reversal trials was only slightly lower than on consistent trials. We discuss this and other alternative models in more detail in *SI Appendix*, section F.

Although an alternative MRW may be found to account for our results, this does not diminish the QRW's significance in highlighting and challenging important assumptions regarding the judgment and decision-making process. In this paper, we have shown that a common assumption of cognitive and neural theories of decision-making—the read-out assumption—is violated even in a simple perceptual task. An interference effect occurred when participants were asked to make a decision about the leftward or rightward motion of a stimulus. Specifically, their subsequent confidence estimates were more conservative than when no earlier decision was made, and they were consequently less overconfident. This result, along with quantitatively superior model fits, lends strong support to the modeling of choice and confidence as a quantum random walk process, a model which describes decision-making as a constructive process wherein a definite state is created from an indefinite superposition. In addition to the cognitive implications, a QRW model of evidence accumulation potentially sidesteps the problem of how a group of neurons can produce observed behavior that is consistent with a single evidence accumulation trajectory (26). The QRW suggests that the mismatch might lie in the cognitive representation of evidence accumulation: instead of treating evidence accumulation as a single trajectory, it may be more accurate to conceptualize it as a wavelike superposition state. In fact, populations of interacting neurons processing evidence in parallel can give rise to a quantum random walk like the one presented here (10), and similar population coding models would certainly be capable of carrying out the necessary operations (27). Hence, quantum random walk theory provides a previously unexamined perspective on the nature of the evidence accumulation process that underlies both cognitive and neural theories of decision-making.

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1. Gold JI, Shadlen MN (2007) The neural basis of decision making. *Annu Rev Neurosci* 30:535–574.
2. Ratcliff R, Smith PL (2004) A comparison of sequential sampling models for two-choice reaction time. *Psychol Rev* 111(2):333–367.
3. Bogacz R, Brown E, Moehlis J, Holmes P, Cohen JD (2006) The physics of optimal decision making: A formal analysis of models of performance in two-alternative forced-choice tasks. *Psychol Rev* 113(4):700–765.
4. Pleskac TJ, Busemeyer JR (2010) Two-stage dynamic signal detection: A theory of choice, decision time, and confidence. *Psychol Rev* 117(3):864–901.
5. Hanes DP, Schall JD (1996) Neural control of voluntary movement initiation. *Science* 274(5286):427–430.
6. Shadlen MN, Newsome WT (2001) Neural basis of a perceptual decision in the parietal cortex (area LIP) of the rhesus monkey. *J Neurophysiol* 86(4):1916–1936.
7. Lichtenstein S, Slovic P, eds (2006) *The Construction of Preference* (Cambridge Univ Press, Cambridge, UK).
8. Busemeyer JR, Wang Z, Townsend JT (2006) Quantum dynamics of human decision-making. *J Math Psychol* 50(3):220–241.
9. Feynman RP, Leighton RB, Sands M (2013) *The Feynman Lectures on Physics, Desktop Edition* (Basic Books, New York), Vol I.
10. Fuss IG, Navarro DJ (2013) Open parallel cooperative and competitive decision processes: A potential provenance for quantum probability decision models. *Top Cogn Sci* 5(4):818–843.
11. Kempe J (2003) Quantum random walks: An introductory overview. *Contemp Phys* 44(4):307–327.
12. Ma WJ, Beck JM, Latham PE, Pouget A (2006) Bayesian inference with probabilistic population codes. *Nat Neurosci* 9(11):1432–1438.
13. Busemeyer JR, Bruza P (2012) *Quantum Models of Cognition and Decision* (Cambridge Univ Press, Cambridge, UK).
14. Resulaj A, Kiani R, Wolpert DM, Shadlen MN (2009) Changes of mind in decision-making. *Nature* 461(7261):263–266.
15. Yu S, Pleskac TJ, Zeigenfuse MD (2015) Dynamics of postdecisional processing of confidence. *J Exp Psychol Gen* 144(2):489–510.
16. Busemeyer JR, Townsend JT (1993) Decision field theory: A dynamic-cognitive approach to decision making in an uncertain environment. *Psychol Rev* 100(3):432–459.
17. Usher M, McClelland JL (2001) The time course of perceptual choice: The leaky, competing accumulator model. *Psychol Rev* 108(3):550–592.
18. Ratcliff R, Rouder JN (1998) Modeling response times for two-choice decisions. *Psychol Sci* 9(5):347–356.
19. Ball K, Sekuler R (1987) Direction-specific improvement in motion discrimination. *Vision Res* 27(6):953–965.
20. Kruschke J (2010) *Doing Bayesian Data Analysis: A Tutorial Introduction with R* (Academic, Waltham, MA).
21. Sniezek JA, Paese PW, Switzer FS, III (1990) The effect of choosing on confidence in choice. *Organ Behav Hum Decis Process* 46(2):264–282.
22. Kass RE, Raftery AE (1995) Bayes factors. *J Am Stat Assoc* 90(430):773–795.
23. Koriat A, Lichtenstein S, Fischhoff B (1980) Reasons for confidence. *J Exp Psychol Learn Mem Cogn* 6(2):107–118.
24. Zylberberg A, Barttfeld P, Sigman M (2012) The construction of confidence in a perceptual decision. *Front Integr Neurosci* 6:79.
25. Green DM, Swets JA (1966) *Signal Detection Theory and Psychophysics* (Wiley, New York), Vol 1.
26. Zandbelt B, Purcell BA, Palmeri TJ, Logan GD, Schall JD (2014) Response times from ensembles of accumulators. *Proc Natl Acad Sci USA* 111(7):2848–2853.
27. Pouget A, Dayan P, Zemel R (2000) Information processing with population codes. *Nat Rev Neurosci* 1(2):125–132.